On the homotopy type of the Beilinson–Drinfeld Grassmannian

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Abstract

Let G be a complex reductive group and let I be a non-empty finite set. The aim of this paper is to prove that an inclusion of open balls $D' \subset D$ on $\mathbb{C}$ induces a stratified homotopy equivalence between the respective Beilinson–Drinfeld Grassmannians $\mathrm{Gr}_{G,D'^I} \hookrightarrow \mathrm{Gr}_{G,D^I}$, a long-standing folklore result. We also prove an analogous result at the level of Ran Grassmannians. We use a purely algebraic approach, showing that automorphisms of algebraic curves can be lifted to automorphisms of the Beilinson–Drinfeld Grassmannian itself. As a consequence, the homotopies appearing in the usual statement can be taken to be stratified isotopies.

We then prove an analogous result for $\mathrm{L}^+G_{\mathbb{C}^I}$, the Beilinson–Drinfeld version of the arc group. We conclude by checking the compatibility between the found isotopies, enhancing the first statement to an equivariant one.

Contents

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1 Introduction

Let G be a complex reductive group and let Gr_G be the affine Grassmannian associated to G. It parametrizes G-torsors on the affine line $\mathbb{A}^1_{\mathbb{C}}$ together with a trivializion at the origin $\{0\} \in \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})$:

$$\mathrm{Gr}_G(R) \simeq \{\mathcal{F} \in \mathrm{Bun}_G(\mathbb{A}^1_R), \alpha \text{ trivialization of } \mathcal{F} \text{ on } \mathbb{A}^1_R \setminus \{0\}_R\}_{/\text{isomorphism}}$$

for any $\mathbb{C}$-algebra R. More generally, given a connected smooth curve X over $\mathbb{C}$ and a finite non-empty set I, the Beilinson–Drinfeld Grassmannian Gr_{G,X^I} (Recall 3.9) is the functor parametrizing

$$\mathrm{Gr}_{G,X^I}(R) \simeq \{x_I \in X^I(R), \mathcal{F} \in \mathrm{Bun}_G(X_R), \alpha \text{ trivialization of } \mathcal{F} \text{ on } X_R \setminus \Gamma_{x_I}\}_{/\text{isomorphism}}$$

where Γ_{x_I} is the union of the graphs of points x_I in X_R (see Notation 3.8). To these algebraic objects, one can associate their analytifications $\mathrm{Gr}^{\mathrm{an}}_G$, $\mathrm{Gr}^{\mathrm{an}}_{G,X^I}$, which consist of their sets of $\mathbb{C}$-points $\mathrm{Gr}_G(\mathbb{C})$ with the complex-analytic topology (see Theorem 2.9 and Theorem 4.1).

Letting I vary in the category $\mathrm{Fin}_{\geq 1,\mathrm{surj}}$ of non-empty finite sets with surjections between them, one can take the colimit of the Gr_{G,X^I}'s in the category of $\mathrm{PSh}(\mathrm{Aff}_{\mathbb{C}})$, and obtain the so-called Ran-Grassmannian $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$. Once more, we can consider its underlying topological space $\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)}$ with the complex-analytic topology, defined as the colimit of the $\mathrm{Gr}^{\mathrm{an}}_{G,X^I}$'s in the category of topological spaces (see Theorem 2.9 and Theorem 4.1).

The spaces $\mathrm{Gr}^{\mathrm{an}}_G$, $\mathrm{Gr}^{\mathrm{an}}_{G,X^I}$ and $\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)}$ carry stratifications, induced by the stratification in Schubert cells of the affine Grassmannian and the so-called incidence stratification of X^I (Recall 3.5 and Recall 3.15). This is formalized in Theorem 2.9 and Theorem 4.1.

1.1 Main results

Consider an open *metric disk* D in $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} = \mathbb{C}$, i.e. an open ball $B(z,r) \subset \mathbb{C}$ centered in $z \in \mathbb{C}$ with radius $r \in \mathbb{R}_{>0}$. Denote by Gr_{G,D^I} the fiber product $\mathrm{Gr}^{\mathrm{an}}_{G,(\mathbb{A}^1_{\mathbb{C}})^I} \times_{\mathbb{C}^I} D^I$ of stratified spaces. In the same way, one defines $\mathrm{Gr}_{G,\mathrm{Ran}(D)}$ to be the restriction of $\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(\mathbb{A}^1_{\mathbb{C}})}$ to $\mathrm{Ran}(D)$.

Our main result concerns the existence of a stratified homotopy equivalence between these spaces (see Definition 4.12).

Theorem A (Corollary 4.14, Corollary 4.15)**.** *Let $D' \subset D \subset \mathbb{C}$ be two metric open disks. The induced open embedding $i : \mathrm{Gr}_{G,D'^I} \hookrightarrow \mathrm{Gr}_{G,D^I}$ is a stratified homotopy equivalence, and the homotopies involved can be taken to be isotopies.*
The same is true for the open embedding $\mathrm{Gr}_{G,\mathrm{Ran}(D')} \hookrightarrow \mathrm{Gr}_{G,\mathrm{Ran}(D)}$.

As a Corollary, we get the following folklore result (see [Lur17, §5.5.4] for notations).

Corollary B (Theorem 4.32, cf. [HY19, Theorem 3.10])**.** *Consider the category* StrTop *of stratified topological spaces. Let W be the family of stratified homotopy equivalences. For any metric disk D,* $\mathrm{Gr}_{G,\mathrm{Ran}(D)}$ *carries a non-unital $\mathbb{E}_2$-algebra structure in* $\mathrm{StrTop}[W^{-1}]$ *with respect to the Cartesian symmetric monoidal structure independent of D.*

The open embeddings in Theorem A actually satisfy an equivariance property. Indeed, let $\mathrm{L}^+G_{X^I}$ be the Beilinson–Drinfeld version of the arc group (Recall 3.27). One again can consider its underlying topological space $\mathrm{L}^+G^{\mathrm{an}}_{X^I}$ with the complex-analytic topology and its restriction $\mathrm{L}^+G_{D^I} \coloneqq \mathrm{L}^+G^{\mathrm{an}}_{X^I} \times_{(X^{\mathrm{an}})^I} D^I$. Given two open metric disks $D' \subset D \subset \mathbb{C}$, we still get that the induced open embedding $i^+ : \mathrm{L}^+G_{D'^I} \hookrightarrow \mathrm{L}^+G_{D^I}$ is a stratified homotopy equivalence, and the homotopies involved can be taken to be isotopies as well (Proposition 4.17).

Theorem C (Theorem 4.18, Theorem 4.19)**.** *Given two metric open disks $D' \subset D \subset \mathbb{C}$, all the mentioned isotopies are compatible with the action of $\mathrm{L}^+G_{D^I}$ on Gr_{G,D^I}. More precisely, there are stratified isotopies $\Psi^{\mathrm{equiv}}_{[0,1]}$ and $\Psi_{[0,1]}$ fitting in*

$$\begin{tikzcd}
{[0,1] \times \left(\mathrm{L}^+G_{\mathbb{C}^I} \times_{\mathbb{C}^I} \mathrm{Gr}_{G,\mathbb{C}^I}\right)} \arrow[r, "\Psi^{\mathrm{equiv}}_{[0,1]}"] \arrow[d] & \mathrm{L}^+G_{\mathbb{C}^I} \times_{\mathbb{C}^I} \mathrm{Gr}_{G,\mathbb{C}^I} \arrow[d] \\
{[0,1] \times \mathrm{Gr}_{G,\mathbb{C}^I}} \arrow[r, "\Psi_{[0,1]}"] & \mathrm{Gr}_{G,\mathbb{C}^I},
\end{tikzcd}$$

which provide stratified isotopies for the diagram

$$\begin{tikzcd}
\mathrm{L}^+G_{D'^I} \times_{D'^I} \mathrm{Gr}_{G,D'^I} \arrow[r, hook, "i^+ \times i"] \arrow[d] & \mathrm{L}^+G_{D^I} \times_{D^I} \mathrm{Gr}_{G,D^I} \arrow[d] \\
\mathrm{Gr}_{G,D'^I} \arrow[r, hook, "i"] & \mathrm{Gr}_{G,D^I}
\end{tikzcd}$$

where the vertical maps are induced by the action. An analogous statement is true at the truncated level (namely for $\mathrm{L}^m G_{D^I} \times_{D^I} \mathrm{Gr}^{(N)}_{G,D^I}$ for any $N \in \mathbb{N}$ and $m \geq m_{N,I}$), and at the Ran level.

1.2 Motivation

To the best of our knowledge, the first time that a statement like Corollary B appeared is in [GL, Remark 9.4.20]. They implicitly compare the $\mathbb{E}_2$-coalgebra structure on cochains $C^*(\mathrm{Gr}_G; \mathbb{Z}_\ell)$ coming from the $\mathbb{E}_2$-structure on Gr_G (a direct consequence of the homotopy equivalence $\mathrm{Gr}_G^{\mathrm{an}} \simeq \Omega G^{\mathrm{an}} = \Omega^2 \mathrm{B}G^{\mathrm{an}}$, see e.g. [Nad03, Theorem 2.1], [PS86, Theorem 8.6.2, 8.6.3]) and the $\mathbb{E}_2$-structure coming from the sheaf

$$\mathcal{A} : \mathrm{Op}(\mathrm{Ran}(\mathbb{C}))^{\mathrm{op}} \to \mathrm{Ch}^*(\mathrm{Mod}_{\mathbb{Z}_\ell})$$

sending an open U of $\mathrm{Ran}(\mathbb{C})$ into $C^*(\mathrm{Gr}_{G,\mathrm{Ran}(\mathbb{C})}|_U; \mathbb{Z}_\ell)$. The reason why the sheaf $\mathcal{A}$ has an $\mathbb{E}_2$-structure is exactly that the functor $\mathrm{Gr}_{G,\mathrm{Ran}(-)}$ has it. This principle constitutes the main motivation for our results.

The first time that Corollary B has been stated explicitly, again to our knowledge, is in [HY19, Theorem 3.10]. The relationship between the present paper and those two results can be summarized as follows.

- Theorem A implies that the inclusion maps $i : \mathrm{Gr}_{G,\mathrm{Ran}(D')} \hookrightarrow \mathrm{Gr}_{G,\mathrm{Ran}(D)}$ induce isomorphisms in cohomology, claimed that appeared in the sketch of proof of [HY19, Proposition 3.17] (which is the main tool used to prove [HY19, Theorem 3.10]). Indeed Theorem A provides a homotopy equivalence between the two spaces, which implies a cohomological equivalence.[1]

- [HY19, Theorem 3.10 and Proposition 3.17] are statements in the *stable* setting, i.e. they concern $\Sigma^\infty_+(\mathrm{Gr}_G^{\mathrm{an}})$, $\Sigma^\infty_+(\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(\mathbb{A}^1_{\mathbb{C}})})$, etc.. In particular, [HY19, Proposition 3.17] says that the map $\Sigma^\infty_+(\mathrm{Gr}_G^{\mathrm{an}}) \to \Sigma^\infty_+(\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(\mathbb{A}^1_{\mathbb{C}})})$ associated to the choice of any point $x \in \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})$ is an equivalence of spectra. The authors then prove that $\Sigma^\infty_+(\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(\mathbb{A}^1_{\mathbb{C}})})$ carries an $\mathbb{E}_2$-structure and therefore this can be transferred to $\Sigma^\infty_+\mathrm{Gr}_G$ via the mentioned equivalence of spectra.

[1] It would be nice to have a purely cohomological argument that does not use a homotopy-theoretic statement.

The present work is a first step in the direction of an *unstable version* of this result, namely that $\mathrm{Gr}_G^{\mathrm{an}}$ admits a non-unital $\mathbb{E}_2$-algebra structure in $\mathrm{StrTop}[W^{-1}]$. Indeed, by Corollary B, $\mathrm{Gr}_{G,\mathrm{Ran}(D)}$ has a non-unital $\mathbb{E}_2$-structure in $\mathrm{StrTop}[W^{-1}]$: so now it remains to inspect the map $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{G,\mathrm{Ran}(D)}$ and prove that the $\mathbb{E}_2$-structure can be transferred to the left-hand-side, in analogy to the stable result.

We conclude by mentioning another motivation for Theorem A, namely Theorem C, its equivariant version: this latter is used in [Noc20] to prove that the ∞-category $\mathcal{C}\mathrm{ons}_{\mathrm{L}^+G^{\mathrm{an}}}(\mathrm{Gr}_G^{\mathrm{an}};\Lambda)$ of $\mathrm{L}^+G^{\mathrm{an}}$-equivariant constructible sheaves with coefficients in Λ-Modules carries an $\mathbb{E}_2$-structure (actually an $\mathbb{E}_3$-structure). Note that, for this application, it is really important to have an unstable statement.

1.3 Outline of the paper

In Section 2 we formalize the fact that the usual analytification functor $(-)^{\mathrm{an}} : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{lft}} \to \mathrm{Top}$ can be enhanced to a functor between the category of *stratified* schemes and *stratified* topological spaces. Moreover we *Kan-extend* it to *small stratified* presheaves and *pro-group stratified* schemes, all locally of finite type (Theorem 2.9).

In Section 3, we present all the needed definitions and properties from the Geometric Langlands area. In particular, we recall the definition of the affine Grassmannian, the Beilinson-Drinfeld Grassmannian and the Ran Grassmannian. Some results are just recollections from other references, some other are folklore properties which we prove in detail. In particular, some classical properties in the Beilinson-Drinfeld setting extend to the Ran setting with nontrivial proof, in that many of them are not abstractly stable under non-filtered colimits.

Section 4 is devoted to the proofs of the main results of the paper. We first observe that for any connected smooth complex curve X there is a morphism of presheaves

$$\underline{\mathrm{Aut}}_{\mathbb{C}}(X) \to \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{Gr}_{G,X^I})$$

lifting an automorphism of X to a (stratified) automorphism of the Beilinson–Drinfeld Grassmannian Gr_{G,X^I} (see Lemma 4.7 and Lemma 4.8.) In particular, one can lift affine transformations $z \mapsto \alpha z+\beta$ in $\mathbb{A}^1_{\mathbb{C}}$. By taking the topological realization (via the above mentioned stratification-preserving analytification functor) of the algebro-geometric objects recalled in Section 3, one can apply this lifting principle to isotopically transform the restrictions Gr_{G,D^I} from any open metric disk D to another. This is also true at the Ran level, i.e. there is a lifting morphism

$$\underline{\mathrm{Aut}}_{\mathbb{C}}(X) \to \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{Gr}_{G,\mathrm{Ran}(X)}).$$

These arguments achieve the proof of Theorem A (see Corollary 4.14 and Corollary 4.15).

Theorem C is proven in the same way: the fact that $\mathrm{L}^+G_{D'^I} \hookrightarrow \mathrm{L}^+G_{D^I}$ and $\mathrm{L}^+G_{\mathrm{Ran}(D')} \hookrightarrow \mathrm{L}^+G_{\mathrm{Ran}(D)}$ are stratified homotopy equivalences follows from a similar lifting principle, and the compatibility with the action follows from the constructions.

Note that some care is needed while establishing some constructions at the Ran level, again because many operations are not abstractly stable under general colimits: see in particular proof of Theorem 4.1.

Finally, we deduce Corollary B from Theorem A by applying Lurie's theorem [Lur17, Theorem 5.4.5.15] saying that non-unital $\mathbb{E}_2$-algebras with values in a symmetric monoidal category $\mathcal{C}^{\times}$ are equivalent to *locally constant* non-unital $\mathrm{Disk}(\mathbb{R}^2)^{\otimes}$-algebras with values in $\mathcal{C}$. Here $\mathrm{Disk}(\mathbb{R}^2)^{\otimes}$ is the operad of topological disks in the real plane, and the local constancy property corresponds to Theorem A.

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2 Stratifications and the analytification functor

The main objects of this paper are the *affine Grassmaniann* Gr_G, the *Beilinson-Drinfeld Grassmannians* Gr_{G,X^I}, the *Ran-Grassmannian* $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$, considered with their respective stratifications. We want to see these objects both from the algebro-geometric and the complex-analytic point of view.

In order to compare the two perspectives, in the present section we will formalize how to analytify stratified schemes and *stratified small presheaves*, in order to obtain stratified topological spaces.

2.1 Stratified small presheaves

Let Y be a topological space. Among the slightly different definitions of *stratification* (see [LWY24] for a full comparison between different definitions) we will stick to the *poset-stratified* one due to its good categorical properties.

Definition 2.1. A *poset-stratified* space is a triple $(Y, P, s : Y \to \mathrm{Alex}(P))$ where

1. Y is a topological space, and P is a poset,
2. $\mathrm{Alex} : \mathrm{Pos} \to \mathrm{Top}$ is the functor associating to a poset P the topological space of elements of P endowed with the Alexandroff topology, and
3. s is a surjective continuous map.

For the sake of notation, we will often use (Y, s) to denote the triple $(Y, P, s : Y \to \mathrm{Alex}(P))$ and we will refer to poset-stratified spaces simply as *stratified spaces.*

A map of stratified spaces is a pair $(f, r) : (Y, s) \to (W, s')$ where $f : Y \to W$ is a continuous map and $r : P \to Q$ is an order-preserving function such that

$$\begin{array}{ccc} Y & \xrightarrow{f} & W \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ \mathrm{Alex}(P) & \xrightarrow{r} & \mathrm{Alex}(Q). \end{array}$$

commutes. We denote by StrTop the category of stratified topological spaces.

Remark 2.2. The category StrTop is complete and cocomplete. Both cocompleteness and completeness are proven in [NL19, Proposition 6.1.4.1]: there, the category considered is the one of *stratified compactly generated spaces* but the proof for StrTop is the same one. Moreover: given a small diagram $A \to \mathrm{StrTop}$, the colimit $\mathrm{colim}_{\alpha\in A}(Y_\alpha, P_\alpha, s_\alpha : Y_\alpha \to \mathrm{Alex}(P_\alpha))$ is

$$(\operatorname*{colim}_{\alpha\in A} Y_\alpha, \operatorname*{colim}_{\alpha\in A} P_\alpha, s : \operatorname*{colim}_{\alpha\in A} Y_\alpha \to \operatorname*{colim}_{\alpha\in A} \mathrm{Alex}(P_\alpha) \to \mathrm{Alex}(\operatorname*{colim}_{\alpha\in A} P_\alpha)).$$

Therefore, the underlying poset (resp. topological space) of the colimit in StrTop is the colimit in Pos of the diagram of underlying posets (resp. topological space in Top).

For limits, the situation is slightly different: indeed in general the underlying topological space will have a finer topology than $\lim_{\alpha\in A} Y_\alpha$ in Top (the underlying poset still coincides with the $\lim_{\alpha\in A} P_\alpha$). Nevertheless for *finite* limits $F \to \mathrm{StrTop}$, we still get that $\lim_{\alpha\in F}(Y_\alpha, P_\alpha, s_\alpha : Y_\alpha \to \mathrm{Alex}(P_\alpha))$ is

$$(\lim_{\alpha\in F} Y_\alpha, \lim_{\alpha\in F} P_\alpha, s : \lim_{\alpha\in F} Y_\alpha \xrightarrow{\lim_{\alpha\in F} s_\alpha} \lim_{\alpha\in F} \mathrm{Alex}(P_\alpha) \xleftarrow{\sim} \mathrm{Alex}(\lim_{\alpha\in F} P_\alpha)),$$

For a proof, one can easily reduce to the case of a finite product; then, one observes that the Alexandroff topology on a product coincides with the box topology, which in turn is the same as the product topology if the product is finite.

If no assumption is made on the shape of the limit, but the posets P_α are all the same P, then $\lim_{\alpha\in A}(Y_\alpha, P, s_\alpha : Y_\alpha \to \mathrm{Alex}(P))$ is

$$(\lim_{\alpha\in A} Y_\alpha, P, s : \lim_{\alpha\in A} Y_\alpha \to \mathrm{Alex}(P)).$$

So also in this case the underlying topological space of the limit of the diagram in StrTop is the limit in Top of the diagram of the underlying topological spaces.

Definition 2.3. If Y is a scheme locally of finite type over $\mathbb{C}$, a stratification of Y is a stratification (Y^{Zar}, s) of the underlying topological space Y^{Zar} with the Zariski topology. The triple $(Y, P, s : Y^{\mathrm{Zar}} \to \mathrm{Alex}(P))$ is called a *stratified scheme* (locally of finite type over $\mathbb{C}$).

A map of stratified schemes (f, r) is a map of scheme f together with an order-preserving function r such that (f^{Zar}, r) is a map of stratified topological spaces.

Let us denote by $\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}$ the category of stratified schemes locally of finite type over $\mathbb{C}$.

Remark 2.4. One can verify, in a manner analogous to the case of StrTop, that the category $\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}$ admits finite limits and they have the form

$$\lim_{\alpha\in F}\Big(Y_\alpha, P_\alpha, s_\alpha : Y_\alpha^{\mathrm{Zar}} \to \mathrm{Alex}(P_\alpha)\Big) = \left(\lim_{\alpha\in F} Y_\alpha, \lim_{\alpha\in F} P_\alpha, s : \left(\lim_{\alpha\in F} Y_\alpha\right)^{\mathrm{Zar}} \to \lim_{\alpha\in F} Y_\alpha^{\mathrm{Zar}} \to \mathrm{Alex}(\lim_{\alpha\in F} P_\alpha)\right).$$

Definition 2.5. Let $\mathcal{C}$ be a locally small category. A *small presheaf* on $\mathcal{C}$ is a small colimit over a diagram of the form $\gamma : A \to \mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ where $\mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ is the Yoneda functor よ. We denote by $\mathrm{PSh}^{\mathrm{small}}(\mathcal{C})$ the full subcategory of $\mathrm{PSh}(\mathcal{C})$ of small presheaves.

A *stratified small presheaf* locally of finite type over $\mathbb{C}$ is then an object of $\mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})$[2].

Remark 2.6. By definition, this is the free cocompletion[3] of $\mathcal{C}$ embedded in it via the Yoneda functor $よ : \mathcal{C} \hookrightarrow \mathrm{PSh}^{\mathrm{small}}(\mathcal{C})$, see [Lin74, Theorem 2.11].

By formal duality, given a locally small category $\mathcal{C}$, the free completion $\mathrm{Fun}^{\mathrm{small}}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ is the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ of small limits of the form $\gamma : A \to \mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ where $\mathcal{C} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ is the co-Yoneda embedding $よ^{\vee}$.

2.2 Stratified analytification

Let us recall the notion of the *analytification functor* from SGA1-XII. For this, let $\mathfrak{L}_{\mathbb{C}}$ be the category of locally $\mathbb{C}$-ringed spaces and let $\mathfrak{An}_{\mathbb{C}}$ the full subcategory of complex analytic spaces inside $\mathfrak{L}_{\mathbb{C}}$.

[2] The category $\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}$ is locally small.

[3] See [EBP21, Definition 4.1] for the definition of free cocompletion of a locally small category.

Theorem 2.7 ([Ray71, Thm. XII.1.1] and [Ray71, §XII.1.2])**.** *Let Y be a scheme locally of finite type over $\mathbb{C}$. Then the functor*

$$\mathrm{Hom}_{\mathfrak{L}_{\mathbb{C}}}(-,Y) : \mathfrak{An}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Set}$$

is representable by a complex analytic space $\mathrm{an}(Y)$*: namely there exists a map of locally $\mathbb{C}$-ringed spaces $\varphi_Y : \mathrm{an}(Y) \to Y$ such that*

$$\mathrm{Hom}_{\mathfrak{An}_{\mathbb{C}}}(T, \mathrm{an}(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{L}_{\mathbb{C}}}(T, Y), \quad f \mapsto \varphi_Y \circ f$$

is a natural bijection (controvariant in T and covariant in Y). Moreover, $\mathrm{an}(Y)$ *coincides, as sets, with $Y(\mathbb{C})$. Denote by Y^{an} the underlying topological space of* $\mathrm{an}(Y)$[4] *(namely, forget the sheaf). This then defines an* analytification functor

$$(-)^{\mathrm{an}} : \mathrm{Sch}_{\mathbb{C}}^{\mathrm{lft}} \to \mathrm{Top}, \quad Y \mapsto Y^{\mathrm{an}}$$

which preserves finite limits.

We now want to enhance and extend this functor to the category of small stratified presheaves $\mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}})$ and then to *pro-group-objects* of stratified schemes. Let us recall the definition of the latter.

Definition 2.8. Let $\mathcal{D}$ be a locally small category: the category of *pro-objects* $\mathrm{Pro}(\mathcal{D})$ of $\mathcal{D}$ is the full subcategory of $\mathrm{Fun}^{\mathrm{small}}(\mathcal{D}, \mathrm{Set})^{\mathrm{op}}$ of small co-filtered limits of representable functors. Let $\mathcal{C}$ be a category with finite products: denote by $\mathrm{Grp}(\mathcal{C})$ the category of group-objects of $\mathcal{C}$. The category of *pro-group-objects* $\mathrm{ProGrp}(\mathcal{C})$ of $\mathcal{C}$ is defined as $\mathrm{Pro}(\mathrm{Grp}(\mathcal{C}))$.

Consider the forgetful functors which forget the datum of a stratification:

$$\mathrm{Fgt}_{\mathrm{str}} : \mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}} \to \mathrm{Sch}_{\mathbb{C}}^{\mathrm{lft}}, \quad \mathrm{Fgt}_{\mathrm{str}} : \mathrm{StrTop} \to \mathrm{Top}.$$

Similarly, consider the forgetful functors which forget the group structure:

$$\mathrm{Fgt}_{\mathrm{grp}} : \mathrm{Grp}(\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}}{}_{/(Y,s)}) \to \mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}}{}_{/(Y,s)}, \quad \mathrm{Fgt}_{\mathrm{grp}} : \mathrm{Grp}(\mathrm{StrTop}_{/(Y,s)^{\mathrm{an}}}) \to \mathrm{StrTop}_{/(Y,s)^{\mathrm{an}}}.$$

Theorem 2.9 (Stratified Analytifications)**.** *The analytification functor from SGA1.XII can be enhanced and extended to*

$$(-)^{\mathrm{an}}_{\mathrm{Str}} : \mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}} \to \mathrm{StrTop}, \quad (-)^{\mathrm{an}}_{\mathrm{PShStr}} : \mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}}) \to \mathrm{StrTop}$$

where the first functor preserves finite limits, the second one preserves small colimits, and the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Sch}_{\mathbb{C}}^{\mathrm{lft}} \arrow[rr, "(-)^{\mathrm{an}}"] & & \mathrm{Top} \\
\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}} \arrow[u, "\mathrm{Fgt}_{\mathrm{str}}"] \arrow[rr, dashed, "(-)^{\mathrm{an}}_{\mathrm{Str}}"] \arrow[d, hook, "\text{よ}"'] & & \mathrm{StrTop} \arrow[u, "\mathrm{Fgt}_{\mathrm{str}}"'] \arrow[d, equal] \\
\mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}}) \arrow[rr, dashed, "(-)^{\mathrm{an}}_{\mathrm{PShStr}}"] & & \mathrm{StrTop}.
\end{tikzcd}$$

[4] This notation differs from the one used in SGA1 [Ray71], where Y^{an} denotes the complex analytic space and not its underlying topological space.

Similarly, for any $(Y,s) \in \mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}$, there are functors

$$(-)^{\mathrm{an}}_{\mathrm{Grp}} : \mathrm{Grp}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}{}_{/(Y,s)}) \to \mathrm{Grp}(\mathrm{StrTop}_{/(Y,s)^{\mathrm{an}}}),$$
$$(-)^{\mathrm{an}}_{\mathrm{ProGrp}} : \mathrm{ProGrp}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}{}_{/(Y,s)}) \to \mathrm{Grp}(\mathrm{StrTop}_{/(Y,s)^{\mathrm{an}}}),$$

which preserves small limits, making the following diagram commute

$$\begin{tikzcd}
\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}{}_{/(Y,s)} \arrow[r, "(-)^{\mathrm{an}}"] & \mathrm{StrTop}_{/(Y,s)^{\mathrm{an}}} \\
\mathrm{Grp}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}{}_{/(Y,s)}) \arrow[u, "\mathrm{Fgt}_{\mathrm{grp}}"] \arrow[r, dashed, "(-)^{\mathrm{an}}_{\mathrm{Grp}}"] \arrow[d, hook, "よ^{\vee}"'] & \mathrm{Grp}(\mathrm{StrTop}_{/(Y,s)^{\mathrm{an}}}) \arrow[u, "\mathrm{Fgt}_{\mathrm{grp}}"'] \arrow[d, equal] \\
\mathrm{ProGrp}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}{}_{/(Y,s)}) \arrow[r, dashed, "(-)^{\mathrm{an}}_{\mathrm{StrGrp}}"] & \mathrm{Grp}(\mathrm{StrTop}_{/(Y,s)^{\mathrm{an}}}).
\end{tikzcd}$$

Proof. Let us first see how to promote the analytification functor to $\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}$. Let $(Y, P, s : Y^{\mathrm{Zar}} \to \mathrm{Alex}(P))$ be an element of $\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}$. The map $\varphi_Y : \mathrm{an}(Y) \to Y$ gives a map of topological spaces $\varphi_Y^{\mathrm{top}} : Y^{\mathrm{an}} \to Y^{\mathrm{Zar}}$. Define s^{an} to be the composite

$$s^{\mathrm{an}} = s \circ \varphi_Y^{\mathrm{top}} : Y^{\mathrm{an}} \to Y^{\mathrm{Zar}} \to \mathrm{Alex}(P).$$

Let $(f,r) : (Y,s) \to (W,s')$ be a stratified map. Consider the map $\mathrm{an}(f) : \mathrm{an}(Y) \to \mathrm{an}(W)$: by definition the map $\mathrm{an}(f)$ fits in the commutative diagram

$$\begin{tikzcd}
\mathrm{an}(Y) \arrow[r, "\mathrm{an}(f)"] \arrow[d, "\varphi_Y"'] & \mathrm{an}(W) \arrow[d, "\varphi_W"] \\
Y \arrow[r, "f"] & W
\end{tikzcd}$$

(by covariance in W). Forgetting the sheaves, we have the commutative diagram

$$\begin{tikzcd}
Y^{\mathrm{an}} \arrow[r, "f^{\mathrm{an}}"] \arrow[d, "\varphi_Y^{\mathrm{top}}"'] & \mathrm{an}(W) \arrow[d, "\varphi_W^{\mathrm{top}}"] \\
Y^{\mathrm{Zar}} \arrow[r, "f^{\mathrm{Zar}}"] \arrow[d, "s"'] & W^{\mathrm{Zar}} \arrow[d, "s'"] \\
\mathrm{Alex}(P) \arrow[r, "r"] & \mathrm{Alex}(Q).
\end{tikzcd}$$

Therefore (f^{an}, r) is a map of stratified spaces $(Y^{\mathrm{an}}, s^{\mathrm{an}}) \to (W^{\mathrm{an}}, s'^{\mathrm{an}})$. This defines a functor

$$(-)^{\mathrm{an}}_{\mathrm{Str}} : \mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}} \to \mathrm{StrTop}, \qquad (Y,s) \mapsto (Y^{\mathrm{an}}, s^{\mathrm{an}}), \text{ and } (f,r) \mapsto (f^{\mathrm{an}}, r),$$

which enhances $(-)^{\mathrm{an}} : \mathrm{Sch}^{\mathrm{lft}}_{\mathbb{C}} \to \mathrm{Top}$. This functor still preserves finite limits. Indeed, take $F \to \mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}$ a finite diagram: then by Remark 2.4

$$\left(\lim_{\alpha \in F}(Y_\alpha, P_\alpha, s_\alpha : Y_\alpha^{\mathrm{Zar}} \to \mathrm{Alex}(P_\alpha))\right)^{\mathrm{an}} = \left(\lim_{\alpha\in F} Y_\alpha, \lim_{\alpha\in F} P_\alpha, s : \left(\lim_{\alpha\in F} Y_\alpha\right)^{\mathrm{Zar}} \to \lim_{\alpha\in F} Y_\alpha^{\mathrm{Zar}} \to \mathrm{Alex}(\lim_{\alpha\in F} P_\alpha)\right)^{\mathrm{an}}.$$

By the definition of $(-)^{\mathrm{an}}_{\mathrm{Str}}$ and by the fact that the original $(-)^{\mathrm{an}}$ preserves finite limits, this in turns is equal to

$$\left(\lim_{\alpha\in F} Y_\alpha^{\mathrm{an}}, \lim_{\alpha\in F} P_\alpha, s^{\mathrm{an}} : \lim_{\alpha\in F} Y_\alpha^{\mathrm{an}} \to \left(\lim_{\alpha\in F} Y_\alpha\right)^{\mathrm{Zar}} \to \lim_{\alpha\in F} Y_\alpha^{\mathrm{Zar}} \to \mathrm{Alex}(\lim_{\alpha\in F} P_\alpha)\right).$$

By the universal property of limits, the map $\lim_{\alpha\in F} Y_\alpha^{\mathrm{an}} \to (\lim_{\alpha\in F} Y_\alpha)^{\mathrm{Zar}} \to \lim_{\alpha\in F} Y_\alpha^{\mathrm{Zar}}$ coincides with the limit map $\lim_{\alpha\in F} Y_\alpha^{\mathrm{an}} \to \lim_{\alpha\in F} Y_\alpha^{\mathrm{Zar}}$. Hence the statement.

Therefore, given a stratified scheme (Y, s), we can consider the slice category on it and get the functor

$$(-)^{\mathrm{an}}_{\mathrm{Str}} : (\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/(Y,s)} \to \mathrm{StrTop}_{/(Y^{\mathrm{an}}, s^{\mathrm{an}})}.$$

Since StrTop is co-complete (see Remark 2.2), then by the universal property of the free co-completion, the left Kan extension exists

$$\begin{tikzcd}
\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}} \arrow[r, "(-)^{\mathrm{an}}_{\mathrm{Str}}"] \arrow[d, hook, "\text{よ}"'] & \mathrm{StrTop} \\
\mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}) \arrow[ur, dashed, "(-)^{\mathrm{an}}_{\mathrm{PShStr}}"'] &
\end{tikzcd}$$

and preserves small colimits (see Remark 2.6). Therefore one automatically gets an analytification functor on $\mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})$.

Let us now focus on the second part of the statement. Since the functor $(-)^{\mathrm{an}}_{\mathrm{Str}} : (\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})_{(Y,s)} \to (\mathrm{StrTop})_{(Y,s)^{\mathrm{an}}}$ preserves finite limits, it upgrades to a functor

$$(-)^{\mathrm{an}}_{\mathrm{Grp}} : \mathrm{Grp}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}{}_{/(Y,s)}) \to \mathrm{Grp}(\mathrm{StrTop}_{/(Y,s)^{\mathrm{an}}}).$$

Since $\mathrm{StrTop}_{/(Y,s)^{\mathrm{an}}}$ is complete, also $\mathrm{Grp}(\mathrm{StrTop}_{/(Y,s)^{\mathrm{an}}})$ is: see [Lur17, Corollary 3.2.2.5][5]. Therefore, as before, by the universal property of the free completion, the right Kan extension exists

$$\begin{tikzcd}
\mathrm{Grp}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}{}_{/(Y,s)}) \arrow[r, "(-)^{\mathrm{an}}_{\mathrm{Grp}}"] \arrow[d, hook, "\text{よ}^{\vee}"'] & \mathrm{Grp}(\mathrm{StrTop}_{/(Y^{\mathrm{an}}, s^{\mathrm{an}})}) \\
\mathrm{Fun}^{\mathrm{small}}(\mathrm{Grp}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}{}_{/(Y,s)}), \mathrm{Set})^{\mathrm{op}} \arrow[ur, dashed] &
\end{tikzcd}$$

and preserves small limits (see Remark 2.6). Since $\mathrm{ProGrp}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}{}_{/(Y,s)})$ is a full subcategory of the free completion, by restriction we have an analytification functor $(-)^{\mathrm{an}}_{\mathrm{ProGrp}}$. $\square$

Notation 2.10. By the sake of notation, in what follows we will usually use $(-)^{\mathrm{an}}$ for any of the previous analytification functors.

3 The Ran Grassmannian as a stratified presheaf

In this Section we recall definitions and properties within the Geometric Langlands needed for the rest of the paper, and we also prove some details/folklore properties, see in particular Recall 3.6, Proposition 3.13, Lemma 3.19, Lemma 3.24, and Section 3.4. Two sources containing very good introductions to the affine Grassmannian and the Beilinson–Drinfeld Grassmannian are [Zhu16] and [BR18]. Other useful properties of the Ran Grassmannian can be found in [Tao20].

[5] In [Lur17, Corollary 3.2.2.5] the statement is about the category of commutative monoids in $\mathrm{StrTop}_{/(Y^{\mathrm{an}}, s^{\mathrm{an}})}$ being complete, from which the case of group objects is easily deduced.

3.1 The stratification on the affine Grassmannian

Here onward, G will be a complex reductive group, and X a smooth (not necessarily proper) connected complex curve.

Notation 3.1. Let R be a $\mathbb{C}$-algebra: $X(R)$ will denote the set of maps $\operatorname{Spec} R \to X$ and X_R will denote the product $X \times_{\mathbb{C}} \operatorname{Spec} R$. We denote by $\mathrm{Aff}_{\mathbb{C}}$ the category of affine $\mathbb{C}$-schemes.

For a scheme Y, $\mathrm{Bun}_G(Y)$ is the groupoid of étale G-torsors over Y. Let us fix $\mathcal{T}_G$ a trivial G-torsor over $\operatorname{Spec}\mathbb{C}$: for any $S \in \mathrm{Sch}_{\mathbb{C}}$ we denote by $\mathcal{T}_{G,S}$ its base change along the structural map $S \to \operatorname{Spec}\mathbb{C}$.

Recall 3.2 (Definition of Gr_G)**.** [Zhu16, (1.2.1)] The *affine Grassmaniann* is the presheaf $\mathrm{Gr}_G : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Set}$ sending

$$\operatorname{Spec} R \mapsto \{(\mathcal{F}, \alpha) : \mathcal{F} \in \mathrm{Bun}_G(\operatorname{Spec} R[\![t]\!]),\ \alpha : \mathcal{F}|_{\operatorname{Spec} R(\!(t)\!)} \xrightarrow{\sim} \mathcal{T}_{G,\operatorname{Spec} R(\!(t)\!)}\}/\sim$$

where $(\mathcal{F}, \alpha) \sim (\mathcal{G}, \beta)$ if and only if there is an isomorphism $\psi : \mathcal{F} \xrightarrow{\sim} \mathcal{G}$ whose restriction makes the following diagram commute

$$\begin{tikzcd}
\mathcal{F}|_{\operatorname{Spec} R(\!(t)\!)} \arrow[rr, "\psi|_{\operatorname{Spec} R(\!(t)\!)}"] \arrow[dr, "\alpha"'] & & \mathcal{G}|_{\operatorname{Spec} R(\!(t)\!)} \arrow[dl, "\beta"] \\
& \mathcal{T}_{G,\operatorname{Spec} R(\!(t)\!)}. &
\end{tikzcd}$$

By [Zhu16, Theorem 1.22], Gr_G is ind-representable by a colimit $\operatorname{colim}_{N\geq 0} \mathrm{Gr}_G^{(N)}$, where each $\mathrm{Gr}_G^{(N)}$ is a projective $\mathbb{C}$-scheme and the transition maps are closed embeddings.

By [Zhu16, Proposition 1.3.6], Gr_G can also be described as the étale sheafification

$$\mathrm{Gr}_G \simeq \left[{}^{\mathrm{L}G}\!/\!_{\mathrm{L}^+G} \right]_{\text{ét}} \tag{3.1}$$

where $\mathrm{L}^+G, \mathrm{L}G$ are étale sheaves in groups defined as

$$\begin{aligned} \mathrm{L}^+G : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} &\to \mathrm{Grp} \\ \operatorname{Spec} R &\mapsto G(R[\![t]\!]), \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{L}G : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} &\to \mathrm{Grp} \\ \operatorname{Spec} R &\mapsto G(R(\!(t)\!)). \end{aligned}$$

By [Zhu16, Proposition 1.3.2], the presheaf L^+G is representable by the inverse limit

$$\mathrm{L}^+G \simeq \lim_{m\geq 0} \mathrm{L}^m G,$$

where $\mathrm{L}^m G$ is the affine group-scheme of finite type over $\mathbb{C}$ representing the functor

$$\begin{aligned} \mathrm{L}^m G : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} &\to \mathrm{Grp} \\ \operatorname{Spec} R &\mapsto G(R[t]/(t^m)). \end{aligned}$$

Fact 3.3. As proven in [Čes24, Theorem 3.4], the quotient presheaf $\mathrm{L}G/\mathrm{L}^+G$ is already an étale sheaf. Indeed every complex reductive group is split[6], hence totally isotropic (see [Čes24, Example 3.2]). Therefore in equation (3.1) we do not need to sheafify.

[6] Every reductive group over a separably closed field is split because it contains a maximal torus [Mil15, (22.23)] and every torus over a separably closed field is split [Mil15, (14.25)].

Thanks to Fact 3.3, the schemes $\mathrm{Gr}_G^{(N)}$ have a very explicit description.

Recall 3.4 (Cartan decomposition)**.** Fix a maximal torus $T \subset \mathrm{GL}_n$ and let $\mathbb{X}_\bullet(T)$ be the group $\mathrm{Hom}(\mathbb{G}_m, T)$ of coweights of T. Fix a set of positive coroots Ψ^+ of T and denote by $\mathbb{X}_\bullet(T)^+$ the set of dominant coweights of T. Endow $\mathbb{X}_\bullet(T)$ by its usual partial order, namely

$$\nu \leq \mu \iff \mu - \nu \in \mathbb{N} \cdot \Psi^+.$$

This restricts to a partial order on $\mathbb{X}_\bullet(T)^+$. Finally fix an embedding of posets $\mathbb{X}_\bullet(T)^+ \hookrightarrow \mathbb{N}^n$. Then one has the identification

$$\begin{aligned}\mathrm{Gr}_{\mathrm{GL}_n}^{(N)}(R) \simeq \{[M] \in \mathrm{GL}_n(R(\!(t)\!))/\mathrm{GL}_n(R[\![t]\!]) : M \text{ has a Cartan decomposition } M = ADB,\\ \text{where } A, B \in \mathrm{GL}_n(R[\![t]\!]) \text{ and } D = \mathrm{diag}(t^{-\nu_1}, \dots, t^{-\nu_n}) \text{ with } 0 \leq \nu_n \leq \cdots \leq \nu_1 \leq N\}.\end{aligned}$$

In the case of an arbitrary G, fix a faithful representation $\rho : G \to \mathrm{GL}_n$ for some n, and this induces a closed embedding $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$ (see [Zhu16, Proposition 1.2.5, 1.2.6]). One then defines the $\mathrm{Gr}_G^{(N)}$'s as the preimage of $\mathrm{Gr}_{\mathrm{GL}_n}^{(N)}$ in Gr_G. Note that ρ also provides an embedding of posets $\mathbb{X}_\bullet(T)^+ \hookrightarrow \mathbb{N}^n$.

Recall 3.5 (Stratification of Gr_G)**.** By [Zhu16, § 2.1, Proposition 2.1.5], the orbits of the action of L^+G on Gr_G by left multiplication are smooth quasi-projective schemes of finite type over $\mathbb{C}$. They are called *Schubert cells* $\mathrm{Gr}_{G,\mu}$ and they are indexed by $\mu \in \mathbb{X}_\bullet(T)^+$.

Given $\mu = (\mu_n \leq \cdots \leq \mu_1) \in \mathbb{X}_\bullet(T)^+$ then

$$\mathrm{Gr}_{\mathrm{GL}_n,\mu}(R) \simeq \{[M] \in \mathrm{Gr}_{\mathrm{GL}_n}(R) : M = ADB, \text{with } A, B \in \mathrm{GL}_n(R[\![t]\!]) \text{ and } D = \mathrm{diag}(t^{-\mu_1}, \dots, t^{-\mu_n})\}.$$

In general, $\mathrm{Gr}_{G,\mu}$ is the preimage of $\mathrm{Gr}_{\mathrm{GL}_n,\mu}$ via the closed embedding $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$ mentioned in Recall 3.4. In particular,

$$\overline{\mathrm{Gr}_{G,\mu}} = \bigcup_{\nu \leq \mu} \mathrm{Gr}_{G,\nu}, \quad \text{and} \quad (\mathrm{Gr}_G^{(N)})_{\mathrm{red}} = \bigcup_{\mu_1 \leq N} \mathrm{Gr}_{G,\mu}.$$

Therefore $\{\mathrm{Gr}_{G,\mu}\}_{\mu \in \mathbb{X}_\bullet(T)^+}$ gives a stratification of Gr_G, making $(\mathrm{Gr}_G, \mathbb{X}_\bullet(T)^+)$ an element of $\mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}})$.

Recall 3.6. The action of $\mathrm{L}^+\mathrm{GL}_n$ on $\mathrm{Gr}_{\mathrm{GL}_n}$ restricts to each $\mathrm{Gr}_{\mathrm{GL}_n}^{(N)}$: indeed the action is a left-multiplication by a matrix with coefficients in $R[\![t]\!]$, so the order of the poles does not increase.

Moreover (left-)multiplication by a matrix of the form $A' + t^N B' \in \mathrm{L}^+\mathrm{GL}_n(R)$, where $A' \in \mathrm{GL}_n(R)$, B' an $n \times n$ matrix with coefficients in R, sends M to $A'MC$ with $C \in \mathrm{GL}_n(R[\![t]\!])$ (because t^N solves the poles in M). Hence the action factors through $\mathrm{GL}_n(R[\![t]\!]/t^N R[\![t]\!]) \simeq \mathrm{GL}_n(R[t]/t^N)$: so we get $\mathrm{L}^N\mathrm{GL}_n \times \mathrm{Gr}_{\mathrm{GL}_n}^{(N)} \to \mathrm{Gr}_{\mathrm{GL}_n}^{(N)}$.

Thanks to the closed embedding $\mathrm{Gr}_G \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_n}$, we recover the above statements for the general case:

$$\forall N \in \mathbb{N},\ \exists m_N : \forall m \geq m_N \quad \mathrm{L}^m G \times \mathrm{Gr}_G^{(N)} \to \mathrm{Gr}_G^{(N)}.$$

Remark 3.7. In general, Gr_G (and $\mathrm{Gr}_G^{(N)}$) is not reduced[7], while the $\mathrm{Gr}_{G,\mu}$ are.

[7] It is reduced, for example, when G is semisimple and simply connected ([Zhu16, Theorem 1.3.11]), but for instance it is not if $G = \mathbb{G}_{\mathrm{m}}$ ([Zhu16, Example 1.3.12]).

3.2 The stratification on the Beilinson–Drinfeld Grassmannian

Denote by $\mathrm{Fin}_{\geq 1,\mathrm{surj}}$ the category of non-empty finite sets with surjective maps between them.

Notation 3.8 (Graphs of points)**.** Let R be a $\mathbb{C}$-algebra, $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$ and $x_I \in X^I(R)$. Let $\mathrm{pr}_i : X^I \to X$ be the projection on the i-th coordinate and denote by x_i the composite $\mathrm{pr}_i \circ x_I$. We denote by Γ_{x_I} the closed (possibly nonreduced) subscheme of X_R corresponding to R-point of $\mathrm{Hilb}_X^{|I|}$ via

$$\operatorname{Spec} R \to X^I \to \mathrm{Sym}_X^{|I|} \simeq \mathrm{Hilb}_X^{|I|}.$$

This subscheme is supported over the union of the graphs Γ_{x_i}. For instance, if $R = \mathbb{C}$, $I = \{1, 2\}$ and $x_1 = x_2$ is a closed point of X, then Γ_{x_I} is the only closed subscheme supported at the point and of length 2.

Recall 3.9 (Definition of Gr_{G,X^I})**.** [Zhu16, §3.1] For any $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$, the *Beilinson–Drinfeld Grassmannian* of power I is the presheaf

$$\begin{gathered}\mathrm{Gr}_{G,X^I} : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Set},\\ \operatorname{Spec} R \mapsto \{(x_I, \mathcal{F}, \alpha) : x_I \in X^I(R),\ \mathcal{F} \in \mathrm{Bun}_G(X_R) \text{ and } \alpha : \mathcal{F}|_{X_R \setminus \Gamma_{x_I}} \xrightarrow{\sim} \mathcal{T}_{G, X_R \setminus \Gamma_{x_I}}\}/\sim,\end{gathered}$$

where $(x_I, \mathcal{F}, \alpha) \sim (y_I, \mathcal{G}, \beta)$ if and only if $x_I = y_I$ in $X^I(R)$ and there is an isomorphism $\psi : \mathcal{F} \xrightarrow{\sim} \mathcal{G}$ whose restriction to $X_R \setminus \Gamma_{x_I}$ makes the following diagram commute:

$$\begin{tikzcd} \mathcal{F}_{X_R \setminus \Gamma_{x_I}} \arrow[rr, "\psi|_{X_R \setminus \Gamma_{x_I}}"] \arrow[dr, "\alpha"'] & & \mathcal{G}_{X_R \setminus \Gamma_{x_I}} \arrow[dl, "\beta"] \\ & \mathcal{T}_{G, X_R \setminus \Gamma_{x_I}}. & \end{tikzcd}$$

As shown in [Zhu16, Theorem 3.1.3], the functor Gr_{G,X^I} is ind-representable by a colimit of projective X^I-schemes $\mathrm{Gr}_{G,X^I}^{(N)}$, and the transition maps are closed embedding.

If $I = \{*\}$, for any point $x_0 : \operatorname{Spec} \mathbb{C} \to X$ we have $\mathrm{Gr}_{G,X} \times_X \{x_0\} \simeq \mathrm{Gr}_G$ ([Zhu16, §3.1]): if $X = \mathbb{A}^1_{\mathbb{C}}$, using the translation automorphism of $\mathbb{A}^1_{\mathbb{C}}$, we get a splitting $\mathrm{Gr}_{G,\mathbb{A}^1_{\mathbb{C}}} \simeq \mathbb{A}^1_{\mathbb{C}} \times \mathrm{Gr}_G$. However, in general no such splitting is guaranteed: what we have instead is that $\mathrm{Gr}_{G,X}$ is isomorphic to a “twisted product”, as we now recall.

Recall 3.10 (Formal coordinates and the torsor $\widehat{X}$)**.** Fix an R-point $x : \operatorname{Spec} R \to X$ and write the map (x, id_R) as an isomorphism η_x followed by a closed embedding

$$\operatorname{Spec} R \xrightarrow{\eta_x} \Gamma_x \overset{i_x}{\hookrightarrow} X_R.$$

Let $\widehat{\mathcal{O}}_{\Gamma_x}$ be the limit of quasi-coherent sheaves of $\mathcal{O}_{X_R}$-algebras $\lim\limits_{n \geq 0} \mathcal{O}_{X_R}/\mathcal{I}^n_{\Gamma_x}$. Then we get a diagram of the form

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "\eta_x", "\sim"'] & \Gamma_x \arrow[r, hook, "i_x"] \arrow[d, hook] & X_R \\ & \underline{\operatorname{Spec}}_{X_R}(\widehat{\mathcal{O}}_{\Gamma_x}) \arrow[ur, "i_{\widehat{x}}"'] & \end{tikzcd}$$

A *formal coordinate at* x is a map $\widehat{x} : \operatorname{Spec} R[\![t]\!] \to X$ such that $\widehat{x}|_{t=0} = x$ and such that it factors as $i_{\widehat{x}} \circ \eta$

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[r, "\eta_x", "\sim"'] \arrow[d, hook] & \Gamma_x \arrow[r, hook, "i_x"] \arrow[d, hook] & X_R \\
\operatorname{Spec} R[\![t]\!] \arrow[r, "\eta", "\sim"'] & \underline{\operatorname{Spec}}_{X_R}(\widehat{\mathcal{O}}_{\Gamma_x}) \arrow[ur, "i_{\widehat{x}}"'] &
\end{tikzcd}$$

where η is an isomorphism. The presheaf of formal coordinates is then defined as

$$\widehat{X} : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Set},$$
$$\operatorname{Spec} R \mapsto \widehat{X}(R) = \{(x, \eta) : x \in X(R), \eta : \operatorname{Spec} R[\![t]\!] \xrightarrow{\sim} \underline{\operatorname{Spec}}_{X_R}(\widehat{\mathcal{O}}_{\Gamma_x}) \text{ such that } \eta|_{t=0} = \eta_x\}.$$

Let $\pi : \widehat{X} \to X$ be the projection $(x, \eta) \mapsto x$. Then we have an action of the ind-group-scheme $\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]$ on it by

$$\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!] \times_X \widehat{X} \to \widehat{X}, \quad (g, x, \eta) \mapsto (x, \eta \circ g).$$

This makes $\widehat{X}$ into a right $\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]$-torsor over X (see [BD05, §5.3.11]).

Recall 3.11 (Twisted product)**.** [Zhu16, §0.3.3]. Consider the right-action of $\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]$ on Gr_G by pull-back, $g \cdot (\mathcal{F}, \alpha) \mapsto (g^*\mathcal{F}, g^*\alpha)$. Given the $\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]$-torsor $\widehat{X}$ and the $\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]$-functor Gr_G, their *twisted product*[8] is

$$\widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]} \mathrm{Gr}_G = \left(\widehat{X} \times \mathrm{Gr}_G / \underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]\right)_{\text{ét}}$$

with $\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]$ acting diagonally.

Remark 3.12. The functor $\widehat{X}$ is an étale torsor. Indeed, the e curve X is étale-locally isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. In this setting X_R is $\operatorname{Spec} R[t]$, the ideal $\mathcal{I}_{\Gamma_x}$ is $(t - r)$, $r \in R$, and thus $\widehat{\mathcal{O}}_{\Gamma_x} \simeq R[\![t]\!]$. Moreover when $X = \mathbb{A}^1_{\mathbb{C}}$ the twisted product $\widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]} \mathrm{Gr}_G$ indeed trivializes as $\mathbb{A}^1_{\mathbb{C}} \times \mathrm{Gr}$. Hence, the twisted product is étale-locally a product $X \times \mathrm{Gr}_G$.

Proposition 3.13. *There is a (noncanonical) isomorphism*

$$\mathrm{Gr}_{G,X} \simeq \widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]} \mathrm{Gr}_G.$$

Proof. Let $x : \operatorname{Spec} R \to X$ be an R-point. Recall that the Beauville-Laszlo theorem [BL95] tells us that the restriction map $\mathrm{Bun}_G(X_R) \to \mathrm{Bun}_G(X_R \setminus \Gamma_x)$ fits in the equivalence of categories

$$\mathrm{Bun}_G(X_R) \simeq \mathrm{Bun}_G(\operatorname{Spec} R[\![t]\!]) \times_{\mathrm{Bun}_G(\operatorname{Spec} R(\!(t)\!))} \mathrm{Bun}_G(X_R \setminus \Gamma_x). \tag{3.2}$$

This induces a morphism of presheaves

$$\widehat{X} \times \mathrm{Gr}_G \to \mathrm{Gr}_X, \qquad [(x, \eta, \widetilde{\mathcal{F}}, \widetilde{\alpha})] \mapsto [(x, \mathcal{F}, \alpha)] \tag{3.3}$$

where $(\mathcal{F}, \alpha)$ is a pair such that

$$\eta^* i_{\widehat{x}}^* \mathcal{F} \simeq \widetilde{\mathcal{F}}, \quad \eta|^*_{\operatorname{Spec} R(\!(t)\!)} i_{\widehat{x}}^* \alpha \simeq \widetilde{\alpha},$$

which is uniquely determined (up to isomorphism) by (3.2). Note that (3.3) is $\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]$-equivariant, because for $[(x, \eta \circ g, g^*\widetilde{\mathcal{F}}, g^*\widetilde{\alpha})]$ the same pair $(\mathcal{F}, \alpha)$ works fine:

$$g^*\widetilde{\mathcal{F}} = g^*(\eta^* i_{\widehat{x}}^* \mathcal{F}), \quad g^*\widetilde{\alpha} = g^*(\eta^* i_{\widehat{x}}^* \alpha).$$

[8] It is also called *contracted product*.

Therefore we get a map of presheaves

$$\widehat{X} \times \mathrm{Gr}_G / \underline{\mathrm{Aut}}_{\mathbb{C}} \mathbb{C}[\![t]\!] \to \mathrm{Gr}_X,$$

which then induces a map between the étale sheaves

$$\widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}} \mathbb{C}[\![t]\!]} \mathrm{Gr}_G \to \mathrm{Gr}_X. \tag{3.4}$$

The map (3.4) is an isomorphism. Indeed, up to passing to an étale chart parametrized by $\mathbb{A}^1_{\mathbb{C}}$, it can be rewritten as the identity map

$$\mathbb{A}^1_{\mathbb{C}} \times \mathrm{Gr} \to \mathbb{A}^1_{\mathbb{C}} \times \mathrm{Gr}$$

(the fact that it is the identity comes from the fact that the identification of $\mathrm{Gr}_{\mathbb{A}^1_{\mathbb{C}}}$ with $\mathbb{A}^1_{\mathbb{C}} \times \mathrm{Gr}$ is exactly the Beauville-Laszlo gluing procedure used in the definition of the map (3.4)). $\square$

Recall 3.14. ([Zhu16, §2.1 and Theorem 1.1.3]) By definition of $\mathrm{Gr}_{G,\nu}$ and $\mathrm{Gr}_G^{(N)}$, the action of $\underline{\mathrm{Aut}}_{\mathbb{C}} \mathbb{C}[\![t]\!]$ on Gr_G restricts to each orbit and to each $\mathrm{Gr}_G^{(N)}$: therefore one can set

$$\mathrm{Gr}_{G,X,\nu} \simeq \widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}} \mathbb{C}[\![t]\!]} \mathrm{Gr}_{G,\nu}, \quad \mathrm{Gr}_{G,X,\leq \mu} \simeq \widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}} \mathbb{C}[\![t]\!]} \mathrm{Gr}_{G,\leq \mu} \quad \text{and} \quad \mathrm{Gr}_{G,X}^{(N)} \simeq \widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}} \mathbb{C}[\![t]\!]} \mathrm{Gr}_G^{(N)}.$$

With this description, it is clear that $\{\mathrm{Gr}_{G,X,\mu}\}_{\mu \leq N}$ are reduced schemes defining stratifications on the $\mathrm{Gr}_{G,X}^{(N)}$'s, which are compatible with the transition maps in N: therefore we have $\mathrm{Gr}_{G,X} \in \mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}_{\mathbb{C}}^{\mathrm{lft}})$.

Recall 3.15 (Stratification of Gr_{G,X^I})**.** ([Nad05, §4.2] and [CvdHS22, 4.3]) Fix $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$ and consider a surjection $\phi : I \twoheadrightarrow J$ of non-empty sets: define then

$$X^\phi = \{x_I \in X^I : x_i = x_j \text{ if and only if } \phi(i) = \phi(j)\},$$

which are locally closed subschemes of X^I. The resulting stratification (X^I, s_I) is known as the *incidence stratification.*

By [Nad05, Proposition 4.2.1], we moreover have an isomorphism

$$\mathrm{Gr}_{G,X^\phi} := \mathrm{Gr}_{G,X^I}|_{X^\phi} \xrightarrow{\sim} \left(\prod_{j=1}^{|J|} \mathrm{Gr}_{G,X} \right)_{\mathrm{disj}}, \tag{3.5}$$

where the right hand side is the open subsheaf of $\prod_{j=1}^{|J|} \mathrm{Gr}_{G,X}$ where the points $(x_1, \dots, x_{|J|}) \in X^{|J|}$ are *distinct.* Isomorphism (3.5) is usually called the *factorization property.* This factorization over X^ϕ restricts to $\mathrm{Gr}_{G,X^I}^{(N)}|_{X^\phi}$ by its definition (see [Zhu16, Thm. 3.1.3]): therefore

$$\mathrm{Gr}_{G,X^I}^{(N)}|_{X^\phi} \simeq \left(\prod_{j=1}^{|J|} \mathrm{Gr}_{G,X}^{(N)} \right)_{\mathrm{disj}}.$$

For any $\underline{\nu} = (\nu^1, \dots, \nu^{|J|}) \in (\mathbb{X}_\bullet(T)^+)^{|J|}$ we get a locally closed subsheaf of Gr_{G,X^ϕ} defined as

$$\mathrm{Gr}_{G,X^\phi,\underline{\nu}} \simeq \left(\prod_{j=1}^{|J|} \mathrm{Gr}_{G,X} \right)_{\mathrm{disj}} \bigcap \prod_{j=1}^{|J|} \mathrm{Gr}_{G,X,\nu^j}. \tag{3.6}$$

Let P_I be the set $\{(\phi : I \twoheadrightarrow J, \underline{\nu})\}_{\phi,\underline{\nu}}$: we say that $(\phi : I \twoheadrightarrow J, \underline{\nu}) \leq (\phi' : I \twoheadrightarrow J', \underline{\nu}')$ if and only if there exists a surjection $\psi : J' \twoheadrightarrow J$ such that $\phi = \psi \circ \phi'$ (so ϕ identifies more coordinates than ϕ') and for every $j \in J$

$$\nu_j \leq \sum_{j' \in \psi^{-1}\{j\}} \nu'_{j'}.$$

Note that for any $(\phi, \underline{\nu}) \in P_I$ we have $\mathrm{Gr}_{G,X^\phi,\underline{\nu}} \subseteq \mathrm{Gr}^{(N)}_{G,X^I}$ for every N big enough. Therefore equation (3.6) defines then a stratification of $\mathrm{Gr}^{(N)}_{G,X^I}$, making $(\mathrm{Gr}_{G,X^I}, P_I)$ an element of $\mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/(X^I,s_I)}$.

With the same proof as the factorization property (3.5), one obtains

Proposition 3.16. *Let $I, J \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$. Let $\left(X^I \times X^J\right)_{\mathrm{disj}}$ be the subscheme of points*

$$(x_1, \ldots, x_{|I|}, y_1, \ldots, y_{|J|})$$

where $\Gamma_{x_i} \cap \Gamma_{y_j} = \varnothing$ whenever $i \in I, j \in J$. There is an isomorphism of stratified ind-schemes

$$(\mathrm{Gr}_{X^I} \times \mathrm{Gr}_{X^J}) \times_{X^I \times X^J} \left(X^I \times X^J\right)_{\mathrm{disj}} \simeq \mathrm{Gr}_{X^{I \sqcup J}} \times_{X^I \times X^J} \left(X^I \times X^J\right)_{\mathrm{disj}}.$$

3.3 The Ran Grassmannian

It is often helpful to combine the ind-schemes Gr_{G,X^I} into one presheaf. Let us start by gluing together the different X^I's.

Definition 3.17 ([Zhu16, Definition 3.3.1])**.** The *Ran presheaf* of X is the functor of unordered non-empty finite sets of *distinct* points on X

$$\begin{aligned} \mathrm{Ran}(X) : \mathrm{Aff}^{\mathrm{op}}_{\mathbb{C}} &\to \mathcal{S}\mathrm{et}, \\ \operatorname{Spec} R &\mapsto \{\underline{x} = \{x_1, \ldots, x_k\} \subset X(R) \text{ non-empty and finite}\}. \end{aligned}$$

This is what is called $\mathrm{Ran}^u(X)$ in [GL, Definition 2.4.2].

Remark 3.18 (Diagonals)**.** For each surjective map $\phi : I \twoheadrightarrow J$ call Δ_ϕ the associated diagonal embedding

$$\Delta_\phi : X^J \hookrightarrow X^I, \qquad x'_J \mapsto x_I \text{ where } x_i = x'_{\phi(i)}.$$

Lemma 3.19. *We have an isomorphism of functors*

$$\mathrm{Ran}(X) \simeq \operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} X^I$$

where the transition maps are the Δ_ϕ's.

Proof. Fix $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$. Consider the *unordering functor*

$$\mathcal{U}_I : X^I \to \mathrm{Ran}(X), \qquad x_I = (x_1, \ldots, x_{|I|}) \mapsto \{x'_1, \ldots, x'_k\}$$

where we forget the order of the x_i's and we do not repeat maps that are equal. So k is the number of different maps in x_I. Notice that for any $\phi : I \twoheadrightarrow J$, $\mathcal{U}_J = \mathcal{U}_I \circ \Delta_\phi$. Hence we get a well-defined surjective map

$$\mathcal{U} : \operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} X^I \to \mathrm{Ran}(X).$$

Let us check that it is injective as well. Suppose that $x_I \in X^I$ and $x_{I'} \in X^{I'}$ are sent to the same $\{x'_1, \dots, x'_k\}$. Fix an order on $\{x'_1, \dots, x'_k\}$: $(x'_1, \dots, x'_{|J|})$ where J has cardinality k. Then define

$$\begin{aligned} \phi : I \twoheadrightarrow J, \qquad & \phi(i) = j \text{ such that } x_i = x'_j \\ \psi : I' \twoheadrightarrow J, \qquad & \phi(i') = j \text{ such that } x_{i'} = x'_j \end{aligned}$$

Then any map $\phi' : I \twoheadrightarrow I'$ such that

$$\phi'(i) = i' \iff \psi \circ \phi'(i) = \psi(i')$$

tells us that x_I and $x_{I'}$ are the same element in the colimit. This proves that the transformation $\mathcal{U}$ is an isomorphism. $\square$

Remark 3.20. Note that $\operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1, \mathrm{surj}}} X^I$ is not a filtered colimit, and in fact one can show that $\mathrm{Ran}(X)$ is not even an étale sheaf (cf. [GL, Warning 2.4]).

Definition 3.21. The *Ran Grassmannian* associated to G and X is the presheaf[9]

$$\begin{gathered} \mathrm{Gr}_{G, \mathrm{Ran}(X)} : \mathrm{Aff}^{\mathrm{op}}_{\mathbb{C}} \to \mathrm{Set}, \\ \operatorname{Spec} R \mapsto \{(\underline{x}, \mathcal{F}, \alpha) : \underline{x} \in \mathrm{Ran}(X)(R), \mathcal{F} \in \mathrm{Bun}_G(X_R), \alpha : \mathcal{F}|_{X_R \setminus \Gamma_{\underline{x}}} \xrightarrow{\sim} \mathcal{T}_{G, X_R \setminus \Gamma_{\underline{x}}}\} / \sim \end{gathered}$$

(where the equivalence relation is the analogous of the one for Gr_{G, X^I} - see Recall 3.9). On morphisms, $\mathrm{Gr}_{G, \mathrm{Ran}(X)}$ sends

$$\operatorname{Spec} S \xrightarrow{f} \operatorname{Spec} R \mapsto [(\underline{x}, \mathcal{F}, \alpha)] \mapsto [(\underline{x} \circ f, (\mathrm{id} \times f)^* \mathcal{F}, (\mathrm{id} \times f)^* \alpha)].$$

Definition 3.22. Define $\delta_\phi : \mathrm{Gr}_{G, X^J} \to \mathrm{Gr}_{G, X^I}$ to be the morphism

$$(x'_J, \mathcal{F}, \alpha) \mapsto (\Delta_\phi(x'_J), \mathcal{F}, \alpha).$$

Note that this definition is well posed since $\Gamma_{x_J} = \Gamma_{\Delta_\phi(x'_J)}$ as closed topological subspaces of X_R.

Lemma 3.23. *The maps δ_ϕ's are stratified.*

Proof. First, it is easy to see that Δ_ϕ is stratified with respect to the incidence stratification. As for δ_ϕ, it sends the stratum $\mathrm{Gr}_{G, X^\psi, \underline{\nu}}$ indexed by $([J \overset{\psi}{\twoheadrightarrow} J'], \underline{\nu} \in (\mathbb{X}_\bullet(T)^+)^{|J'|})$ of Gr_{G, X^J} into the stratum $\mathrm{Gr}_{G, X^{\psi \circ \phi}, \underline{\nu}}$ indexed by $([I \overset{\psi \circ \phi}{\twoheadrightarrow} J'], \underline{\nu} \in (\mathbb{X}_\bullet(T)^+)^{|J'|})$ of Gr_{G, X^I}. $\square$

Lemma 3.24. *For each $I \in \mathrm{Fin}_{\geq 1, \mathrm{surj}}$,*

$$\mathrm{Gr}_{G, X^I} \simeq X^I \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{G, \mathrm{Ran}(X)}$$

where the map $X^I \to \mathrm{Ran}(X)$ is $\mathcal{U}_I$. Moreover, there is an isomorphism of presheaves

$$\mathrm{Gr}_{G, \mathrm{Ran}(X)} \simeq \operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1, \mathrm{surj}}} \mathrm{Gr}_{G, X^I}$$

where the transition maps in the colimit are the δ_ϕ's.

[9]Other versions of the Ran Grassmannian are considered in [GL, Definition 3.2.3]. If π_0 denotes the functor

$$\mathrm{Fun}(\mathrm{Aff}^{\mathrm{op}}_{\mathbb{C}}, \mathrm{groupoids}) \to \mathrm{Fun}(\mathrm{Aff}^{\mathrm{op}}_{\mathbb{C}}, \mathrm{Set})$$

induced by $\pi_0 : \{\mathrm{groupoids}\} \to \mathrm{Set}$, then

$$\mathrm{Gr}_{G, \mathrm{Ran}(X)} \simeq \pi_0 \mathrm{Ran}^u_G(X)$$

where the right-hand-side is in the notations of *loc.cit.*.

Proof. The first part follows directly from the definition, since for any $x_I \in X^I(R)$, Γ_{x_I} only depends on $\mathcal{U}_I(x_I)$.

Therefore, by universality of colimits, we get

$$\operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \mathrm{Gr}_{G,X^I} \simeq \operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \left(X^I \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{G,\mathrm{Ran}(X)}\right) \simeq \left(\operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} X^I\right) \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{G,\mathrm{Ran}(X)}$$

which is isomorphic to $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$ by Lemma 3.19.

The second part of the statement follows straightforwardly by looking at Definition 3.22: the essential point is that the definition of Γ_{x_I} only depends on image of x_I under the map $\mathcal{U} : X^I \to \mathrm{Ran}(X)$. □

Remark 3.25 (Stratification of $\mathrm{Ran}(X)$ and of $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$)**.** Lemma 3.19, Lemma 3.23 and Lemma 3.24 allow to endow $\mathrm{Ran}(X)$ and $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$ with a stratification. Indeed, one can view the two colimits

$$\operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} X^I, \qquad \operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \mathrm{Gr}_{G,X^I} = \operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \operatorname*{colim}_{N\in\mathbb{N}} \mathrm{Gr}^{(N)}_{G,X^I} \tag{3.7}$$

as colimits in $\mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})$, because $X^I, \mathrm{Gr}^{(N)}_{G,X^I}$'s are objects of $\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}$ and Δ_ϕ, δ_ϕ's are stratified maps. More precisely, the forgetful functor $\mathrm{Fgt}_{\mathrm{str}} : \mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}} \to \mathrm{Sch}^{\mathrm{lft}}_{\mathbb{C}} \overset{よ}{\hookrightarrow} \mathrm{PSh}^{\mathrm{small}}(\mathrm{Sch}^{\mathrm{lft}}_{\mathbb{C}})$ that forgets the stratification can be left-Kan-extended to a colimit-preserving forgetful functor

$$\mathrm{Fgt}_{\mathrm{str}} : \mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}) \to \mathrm{PSh}^{\mathrm{small}}(\mathrm{Sch}^{\mathrm{lft}}_{\mathbb{C}}).$$

We hence obtain that

$$\mathrm{Fgt}_{\mathrm{str}}\left(\operatorname*{colim}^{\mathrm{PSh}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} X^I\right) \simeq \mathrm{Ran}(X), \text{ and } \mathrm{Fgt}_{\mathrm{str}}\left(\operatorname*{colim}^{\mathrm{PSh}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \mathrm{Gr}_{G,X^I}\right) \simeq \mathrm{Gr}_{G,\mathrm{Ran}(X)}. \tag{3.8}$$

By abuse of notation, from now onwards, $\mathrm{Ran}(X)$ and $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$ will always be understood as objects in $\mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})$, i.e. as the arguments between brackets in (3.8).

3.4 The action of $\mathrm{L}^+G_{\mathrm{Ran}(X)}$ on $\mathbf{Gr}_{G,\mathbf{Ran}(X)}$

Notice that the action $\mathrm{L}^+G \times \mathrm{Gr}_G \to \mathrm{Gr}_G$ is a stratified map once one endows L^+G with the trivial stratification.

After recalling the generalization of L^+G to the Beilison–Drinfeld setting and then to the Ran setting, we will see that also in these cases one gets *stratifed* actions.

Recall 3.26 (Infinitesimal formal neighbourhood)**.** Given $x_I \in X^I(R)$, denote by $\widehat{\mathcal{O}}_{\Gamma_{x_I}}$ the sheaf of rings $\lim\limits_{n\geq 0} \mathcal{O}_{X_R}/\mathcal{I}^n_{\Gamma_{x_I}}$. Recall that this limit does not depend on the scheme structure of the closed Γ_{x_I} but only on its topology. Denote by $\widetilde{\Gamma}_{x_I}$ the relative spectrum $\underline{\mathrm{Spec}}_{X_R}(\widehat{\mathcal{O}}_{\Gamma_{x_I}})$:

$$\begin{tikzcd}
\Gamma_{x_I} \arrow[r, hook, "i_{x_I}"] \arrow[d, hook] & X_R. \\
\widetilde{\Gamma}_{x_I} = \underline{\mathrm{Spec}}_{X_R}(\widehat{\mathcal{O}}_{\Gamma_{x_I}}) \arrow[ur, "i_{\widehat{x_I}}"'] &
\end{tikzcd}$$

In the case of $I = *$ this recovers the map $i_{\widehat{x}}$ and the scheme $\widetilde{\Gamma}_x = \underline{\mathrm{Spec}}_{X_R}(\widehat{\mathcal{O}}_{\Gamma_x})$ of Recall 3.10. Hence $\widetilde{\Gamma}_{x_I}$ can be viewed as an *infinitesimal formal neighborhood* of Γ_{x_I}. By abuse of notation, we will denote by $i_{\widehat{x_I}}$ also its restriction to the open $\widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_I}$.

Recall 3.27 (Beilinson-Drinfeld version of L^+G)**.** For $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$, define

$$\mathrm{L}^+G_{X^I} : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Set}, \qquad \operatorname{Spec} R \mapsto \{(x_I, g) :\, x_I \in X^I(R), g \in G(\widetilde{\Gamma}_{x_I})\}.$$

Note that $G(\widetilde{\Gamma}_{x_I}) \simeq \underline{\mathrm{Aut}}(\mathcal{T}_{G,\widetilde{\Gamma}_{x_I}})$ (because any G-equivariant automorphism $\mathcal{T}_{G,\widetilde{\Gamma}_{x_I}} \simeq G \times \widetilde{\Gamma}_{x_I} \to G \times \widetilde{\Gamma}_{x_I}$ over $\widetilde{\Gamma}_{x_I}$ is determined by $\{e_G\} \times \widetilde{\Gamma}_{x_I} \to G$).

Remark 3.28. Let $I = *$, $X = \mathbb{A}^1_{\mathbb{C}}$ and consider the point $0 : \operatorname{Spec} \mathbb{C} \to \mathbb{A}^1_{\mathbb{C}}$. Since $R[\![t]\!] \simeq \widehat{\mathcal{O}}_{\Gamma_0}$ then $\mathrm{Aut}(\mathcal{T}_{G,\widetilde{\Gamma}_0}) \simeq \mathrm{Aut}(\mathcal{T}_{G,\operatorname{Spec} R[\![t]\!]})$ and $\mathrm{L}^+G_{\mathbb{A}^1_{\mathbb{C}}}|_0 \simeq \mathrm{L}^+G$.

Remark 3.29. Consider

$$\mathrm{L}^mG_{X^I} : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Set}, \qquad \operatorname{Spec} R \mapsto \{(x_I, g) : x_I \in X^I(R), g \in G(\Gamma^m_{x_I})\}$$

where $\Gamma^m_{x_I}$ is a short-hand for $\operatorname{Spec}_{X_R} \mathcal{O}_{X_R}/\mathcal{I}^m_{\Gamma_{x_I}}$. These are smooth group X^I-schemes and there is an isomorphism

$$\mathrm{L}^+G_{X^I} \simeq \lim_{m\geq 0} \mathrm{L}^mG_{X^I}$$

(see [Ras18, Lemma 2.5.1]). Consider the forgetful functor $\mathrm{L}^mG_{X^I} \to X^I$: pulling back the incidence stratification on X^I, we give a stratification to $\mathrm{L}^mG_{X^I}$, making $\mathrm{L}^+G_{X^I}$ an element of $\mathrm{ProGrp}\left(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}\,/(X^I,s_I)}\right)$.

Definition 3.30 (Ran version of L^+G)**.** Define

$$\mathrm{L}^+G_{\mathrm{Ran}(X)} : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Set}, \qquad \operatorname{Spec} R \mapsto \{(\underline{x}, g) : \underline{x} \in \mathrm{Ran}(X)(R), g \in G(\widetilde{\Gamma}_{\underline{x}})\}.$$

This is well defined because as said before the scheme $\widehat{\mathcal{O}}_{\Gamma_{\underline{x}}}$ depends neither on the order of the points nor on the schematic structure of $\Gamma_{\underline{x}}$ (only on its topology).

Lemma 3.31. *For any $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$,*

$$\mathrm{L}^+G_{X^I} \simeq X^I \times_{\mathrm{Ran}(X)} \mathrm{L}^+G_{\mathrm{Ran}(X)}.$$

Moreover, there is an isomorphism of presheaves

$$\mathrm{L}^+G_{\mathrm{Ran}(X)} \simeq \operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \mathrm{L}^+G_{X^I},$$

where transition maps are $\delta^{\mathrm{grp}}_{\phi} : (x_I, g) \mapsto (\Delta_\phi(x_I), g)$.

Proof. The first part follows from the definitions, since for any $x_I \in X^I(R)$, $\widetilde{\Gamma}_{x_I}$ only depends on $\mathcal{U}_I(x_I)$. The rest of the proof is analogous to the proof of Lemma 3.24. ☐

Proposition 3.32. *The complex presheaf $\mathrm{L}^+G_{\mathrm{Ran}(X)}$ can be promoted to an element of*

$$\mathrm{Grp}\left(\mathrm{PSh}^{\mathrm{small}}(\mathrm{ProStrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/(\mathrm{Ran}(X),s_{\mathrm{Ran}})}\right).$$

Proof. Recall that $\mathrm{L}^+G_{X^I}$ is a pro-group object in $\mathrm{ProGrp}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}\,/(X^I,s_I)})$. Forgetting the group structure, it can be viewed as an element of $(\mathrm{ProStrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/(X^I,s_I)} \overset{\text{よ}}{\hookrightarrow} \mathrm{PSh}^{\mathrm{small}}(\mathrm{ProStrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/(X^I,s_I)}$. Considering the map

$$\mathrm{L}^+G_{X^I} \to X^I \to \mathrm{Ran}(X)$$

we actually have that $\mathrm{L}^+G_{X^I} \in \mathrm{PSh}^{\mathrm{small}}(\mathrm{ProStrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/(\mathrm{Ran}(X),s_{\mathrm{Ran}})}$. Therefore, with the same argument done in Remark 3.25, by Lemma 3.31 the functor $\mathrm{L}^+G_{\mathrm{Ran}(X)}$ can be seen as an object of $\mathrm{PSh}^{\mathrm{small}}(\mathrm{ProStrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/(\mathrm{Ran}(X),s_{\mathrm{Ran}})}$.

Moreover, there is a natural "relative multiplication" given by

$$\mathrm{L}^+G_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)} \mathrm{L}^+G_{\mathrm{Ran}(X)}, \quad (\underline{x}, g).(\underline{x}, h) \mapsto (\underline{x}, gh). \tag{3.9}$$

This does not quite make $\mathrm{L}^+G_{\mathrm{Ran}(X)}$ into an object of $\mathrm{Grp}\left(\mathrm{PSh}^{\mathrm{small}}(\mathrm{ProStrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/\mathrm{Ran}(X)}\right)$, in that we need to upgrade this multiplication structure to the stratified level. To do so, we present it in a different way: namely, combine Lemma 3.31

$$\operatorname*{colim}_{I\in\mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \left(\mathrm{L}^+G_{X^I} \times_{X^I} \mathrm{L}^+G_{X^I}\right) \simeq \operatorname*{colim}_{I\in\mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \left((\mathrm{L}^+G_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)} X^I) \times_{X^I} (\mathrm{L}^+G_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)} X^I)\right)$$

with universality of colimits in $\mathrm{PSh}(\mathrm{ProStrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/(\mathrm{Ran}(X),s_{\mathrm{Ran}})}$

$$\operatorname*{colim}_{I} \left(X^I \times_{\mathrm{Ran}(X)} \left(\mathrm{L}^+G_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)} \mathrm{L}^+G_{\mathrm{Ran}(X)}\right)\right) \simeq \mathrm{L}^+G_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)} \mathrm{L}^+G_{\mathrm{Ran}(X)}. \tag{3.10}$$

to get an isomorphism

$$\mathrm{L}^+G_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)} \mathrm{L}^+G_{\mathrm{Ran}(X)} \simeq \operatorname*{colim}_{I\in\mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \left(\mathrm{L}^+G_{X^I} \times_{X^I} \mathrm{L}^+G_{X^I}\right)$$

in the category $\mathrm{PSh}(\mathrm{ProStrSch}^{\mathrm{lft}}_{\mathbb{C}})_{(\mathrm{Ran}(X),s_{\mathrm{Ran}})}$. In this way, the multiplication law (3.9) can be presented as arising from the multiplication law of $\mathrm{L}^+G_{X^I}$ by passing to colimits, and hence inherites the wanted lift to the stratified setting. $\square$

Definition 3.33 (Definition of $\mathrm{Gr}^{\mathrm{loc}}_{G,\mathrm{Ran}(X)}$)**.** For $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$, we denote by $\mathrm{Gr}^{\mathrm{loc}}_{G,X^I}$ the presheaf

$$\begin{gathered}\mathrm{Gr}^{\mathrm{loc}}_{G,X^I} : \mathrm{Aff}^{\mathrm{op}}_{\mathbb{C}} \to \mathrm{Set},\\ \operatorname{Spec} R \mapsto \{(x_I, \widetilde{\mathcal{F}}, \widetilde{\alpha}) : x_I \in X^I(R), \widetilde{\mathcal{F}} \in \mathrm{Bun}_G(\widetilde{\Gamma}_{x_I}), \widetilde{\alpha} : \widetilde{\mathcal{F}}|_{\widetilde{\Gamma}_{x_I}\setminus\Gamma_{x_I}} \xrightarrow{\sim} \mathcal{T}_{G,\widetilde{\Gamma}_{x_I}\setminus\Gamma_{x_I}}\}/\sim\end{gathered}$$

(where the equivalence relation is the analogue of the one for Gr_{G,X^I} - see Recall 3.9).

Analogously, define the presheaf

$$\begin{gathered}\mathrm{Gr}^{\mathrm{loc}}_{G,\mathrm{Ran}(X)} : \mathrm{Aff}^{\mathrm{op}}_{\mathbb{C}} \to \mathrm{Set},\\ \operatorname{Spec} R \mapsto \{(\underline{x}, \widetilde{\mathcal{F}}, \widetilde{\alpha}) : \underline{x} \in \mathrm{Ran}(X)(R), \widetilde{\mathcal{F}} \in \mathrm{Bun}_G(\widetilde{\Gamma}_{\underline{x}}), \widetilde{\alpha} : \widetilde{\mathcal{F}}|_{\widetilde{\Gamma}_{\underline{x}}\setminus\Gamma_{\underline{x}}} \xrightarrow{\sim} \mathcal{T}_{G,\widetilde{\Gamma}_{\underline{x}}\setminus\Gamma_{\underline{x}}}\}/\sim\end{gathered}$$

(where the equivalence relation is the analogue of the one for Gr_{G,X^I} - see Recall 3.9).

Lemma 3.34. *There is an isomorphism of presheaves*

$$\mathrm{Gr}^{\mathrm{loc}}_{G,\mathrm{Ran}(X)} \simeq \operatorname*{colim}_{I\in\mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \mathrm{Gr}^{\mathrm{loc}}_{G,X^I},$$

where the transition maps are $\delta^{\mathrm{loc}}_\phi : (x_I, \widetilde{\mathcal{F}}, \widetilde{\alpha}) \mapsto (\Delta_\phi(x_I), \widetilde{\mathcal{F}}, \widetilde{\alpha})$.

Proof. Analogous to the proof of Lemma 3.24. $\square$

Lemma 3.35. *The restriction map*

$$\mathrm{Gr}_{G,X^I} \to \mathrm{Gr}^{\mathrm{loc}}_{G,X^I}, \qquad (\underline{x}, \mathcal{F}, \alpha) \mapsto (\underline{x}, i^*_{\hat{\underline{x}}}\mathcal{F}, i^*_{\hat{\underline{x}}}\alpha)$$

is an equivalence of presheaves. Moreover, these maps respect $\delta_\phi, \delta^{\mathrm{loc}}_\phi$'s and hence glue to an equivalence of presheaves $\mathrm{Gr}_{G,\mathrm{Ran}(X)} \to \mathrm{Gr}^{\mathrm{loc}}_{G,\mathrm{Ran}(X)}$.

Proof. Since these maps commute with the $\delta_\phi, \delta^{\mathrm{loc}}_\phi$'s, it suffices to prove the statement for Gr_{G,X^I}. Furthermore, since the restriction map commutes with the forgetful functor towards X^I, it is enough to check it is an equivalence on fibers. So let us fix $x_I \in X^I(R)$ and compare the two fibers

$$\{\mathcal{F} \in \mathrm{Bun}_G(X_R), \alpha : \mathcal{F}|_{X_R \setminus \Gamma_{x_I}} \xrightarrow{\sim} \mathcal{T}_{G,\, X_R \setminus \Gamma_{x_I}}\}_{/\sim},$$
$$\{\widetilde{\mathcal{F}} \in \mathrm{Bun}_G(\widetilde{\Gamma}_{x_I}), \widetilde{\alpha} : \widetilde{\mathcal{F}}|_{\widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_I}} \xrightarrow{\sim} \mathcal{T}_{G,\, \widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_I}}\}_{/\sim}.$$

Now, the restriction map at the level of fibers coincides with taking the π_0 of the map of groupoids

$$\mathrm{Bun}_G(X_R) \times_{\mathrm{Bun}_G(X_R \setminus \Gamma_{x_I})} \{\mathcal{T}_{G,\, X_R \setminus \Gamma_{x_I}}\} \to \mathrm{Bun}_G(\widetilde{\Gamma}_{x_I}) \times_{\mathrm{Bun}_G(\widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_I})} \{\mathcal{T}_{G,\, \widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_I}}\}, \tag{3.11}$$

where the restriction is again given by $\widehat{x}_I : \widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_I} \to X_R \setminus \Gamma_{x_I}$.

It thus suffices to show that the map at the level of groupoids is an equivalence: this is exactly the "family" version of the Beauville-Laszlo theorem [BD05, Remark 2.3.7]. Indeed, the restriction map gives an equivalence

$$\mathrm{Bun}_G(X_R) \times_{\mathrm{Bun}_G(X_R \setminus \Gamma_{x_I})} \{\mathcal{T}_{G,\, X_R \setminus \Gamma_{x_I}}\} \simeq$$
$$\mathrm{Bun}_G(\widetilde{\Gamma}_{x_I}) \times_{\mathrm{Bun}_G(\widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_I})} \mathrm{Bun}_G(X_R \setminus \Gamma_{x_I}) \times_{\mathrm{Bun}_G(X_R \setminus \Gamma_{x_I})} \{\mathcal{T}_{G,\, X_R \setminus \Gamma_{x_I}}\}$$

which is in turn equivalent to the right-hand side

$$\mathrm{Bun}_G(\widetilde{\Gamma}_{x_I}) \times_{\mathrm{Bun}_G(\widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_I})} \{\mathcal{T}_{G,\, \widetilde{\Gamma}_{x_I} \setminus \Gamma_{x_I}}\}$$

$\square$

Remark 3.36. In particular the functor $\mathrm{Gr}^{\mathrm{loc}}_{G,X}$ is an étale sheaf. Furthermore, for $I = *$, it is canonically isomorphic to the twisted product $\widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]} \mathrm{Gr}_G$. Indeed pick an affine étale cover of X made of $\mathbb{A}^1_{\mathbb{C}}$: over the affine line the two descriptions are the same via

$$(x, \eta, \widetilde{\mathcal{F}}, \widetilde{\alpha}) \mapsto (x, (\eta^{-1})^* \widetilde{\mathcal{F}}, (\eta^{-1})^* \widetilde{\alpha}).$$

Remark 3.37. The functor $\mathrm{L}^+ G_{X^I}$ acts on $\mathrm{Gr}^{\mathrm{loc}}_{G,X^I}$ over X^I by modification of the trivialization $\widetilde{\alpha} \mapsto g|_{\tilde{\Gamma}_{x_I} \setminus \Gamma_{x_I}} \circ \widetilde{\alpha}$. By Lemma 3.35, we get an induced action over X^I

$$\mathrm{L}^+ G_{X^I} \times_{X^I} \mathrm{Gr}_{G,X^I} \to \mathrm{Gr}_{G,X^I}. \tag{3.12}$$

Proposition 3.38. *The action* (3.12) *is stratified. Moreover there exists an integer $m_{N,I}$ such that for any $m \geq m_{N,I}$ action* (3.12) *factors as a stratified action over X^I*

$$\mathrm{L}^m G_{X^I} \times_{X^I} \mathrm{Gr}^{(N)}_{G,X^I} \to \mathrm{Gr}^{(N)}_{G,X^I}$$

Proof. First restrict the action to X^ϕ, $\phi : I \twoheadrightarrow J$: by factorization property (3.5) we get

$$\mathrm{L}^+G_{X^I}|_{X^\phi} \times_{X^\phi} \left(\prod_{j=1}^{|J|} \mathrm{Gr}_{G,X} \right)^\circ \to \mathrm{Gr}_{G,X^I}.$$

Notice that the restriction map $\mathrm{Gr}_{G,X^I} \to \mathrm{Gr}^{\mathrm{loc}}_{G,X^I}$ sends the open $\left(\prod_{i=1}^{|J|} \mathrm{Gr}_{G,X}\right)^\circ$ in $\left(\prod_{i=1}^{|J|} \mathrm{Gr}^{\mathrm{loc}}_{G,X}\right)^\circ$ which is equal to $\left(\prod_{i=1}^{|J|} \widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]} \mathrm{Gr}_G\right)^\circ$ by Remark 3.36. Since the action of L^+G on Gr_G is stratified (by definition of the stratification on Gr_G), then

$$\mathrm{L}^+G_{X^I}|_{X^\phi} \times_{X^\phi} \left(\prod_{i=1}^{|J|} \widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]} \mathrm{Gr}_G \right)^\circ \to \left(\prod_{i=1}^{|J|} \widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]} \mathrm{Gr}_G \right)^\circ.$$

One conclude noticing that the isomorphism $\mathrm{Gr}^{\mathrm{loc}}_{G,X} = \widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]} \mathrm{Gr}_G \to \mathrm{Gr}_{G,X}$ is the inverse of the restriction morphism (see the proof of Proposition 3.13).

By the same argument, this implies that the restriction map is compatible with the filtration $\mathrm{Gr}^{(N)}_{G,X^I}$. Furthermore, the action restricted to $\mathrm{Gr}^{(N)}_{G,X^I}$ factors through the quotient $\mathrm{L}^+G_{X^I} \twoheadrightarrow \mathrm{L}^mG_{X^I}$ for some number $m_{N,I}$[10], see the proof of [Ric14, Corollary 2.7]. □

Corollary 3.39. *We have a stratified action of the group presheaf* $\mathrm{L}^+G_{\mathrm{Ran}(X)}$ *on* $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$ *over* $\mathrm{Ran}(X)$*:*

$$\mathrm{L}^+G_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)} \mathrm{Gr}_{G,\mathrm{Ran}(X)} \to \mathrm{Gr}_{G,\mathrm{Ran}(X)}.$$

Proof. Note that $\delta_\phi, \delta^{\mathrm{loc}}_\phi$ and $\delta^{\mathrm{grp}}_\phi$ do not change the component $x_I \in X^I(R)$. By Lemma 3.35 we have that the actions (3.12) over the X^I's are compatible with each other. Hence the statement. □

4 Isotopy invariance

4.1 Topological realization

In the previous section we recalled the objects of interest from the area of Geometric Langlands, namely the affine Grassmannian, the Beilinson–Drinfeld Grassmannian, the Ran Grassmannian and the arc group (together with its Beilinson–Drinfeld and Ran versions). As stated in the Introduction, we are interest also in their complex-analytic counterpart. The machinery built in Section 2 allows us to take their analytification without losing the information about their stratifications and stratified group actions.

Theorem 4.1. *Applying Theorem 2.9, we get the following objects*

$$\mathrm{L}^mG^{\mathrm{an}}, \mathrm{L}^+G^{\mathrm{an}} \in \mathrm{Grp}(\mathrm{StrTop}), \quad \mathrm{L}^mG^{\mathrm{an}}_{X^I}, \mathrm{L}^+G^{\mathrm{an}}_{X^I} \in \mathrm{Grp}(\mathrm{StrTop}_{/(X^{\mathrm{an}})^I}),$$
$$(\mathrm{Gr}^{(N)}_G)^{\mathrm{an}}, \mathrm{Gr}^{\mathrm{an}}_G \in \mathrm{StrTop}, \quad (\mathrm{Gr}^{(N)}_{G,X^I})^{\mathrm{an}}, \mathrm{Gr}^{\mathrm{an}}_{G,X^I} \in \mathrm{StrTop}_{/(X^{\mathrm{an}})^I},$$
$$\mathrm{Ran}(X)^{\mathrm{an}}, \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)} \in \mathrm{StrTop}, \quad \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(X)} \in \mathrm{Grp}(\mathrm{StrTop}_{/\mathrm{Ran}(X)^{\mathrm{an}}})$$

together with:

- *stratified actions*

$$\forall\, m \geq m_N \quad \mathrm{L}^mG^{\mathrm{an}} \times (\mathrm{Gr}^{(N)}_G)^{\mathrm{an}} \to (\mathrm{Gr}^{(N)}_G)^{\mathrm{an}} \quad \text{and} \quad \mathrm{L}^+G^{\mathrm{an}} \times \mathrm{Gr}^{\mathrm{an}}_G \to \mathrm{Gr}^{\mathrm{an}}_G,$$

[10] From the proof of [Ric14, Corollary 2.7] one can see that $m_{N,I}$ depends on N but not on I.

- *stratified actions over* X^I

$$\forall m \geq m_{N,I} \quad \mathrm{L}^m G^{\mathrm{an}}_{X^I} \times_{X^{\mathrm{an}\,I}} (\mathrm{Gr}^{(N)}_{G,X^I})^{\mathrm{an}} \to (\mathrm{Gr}^{(N)}_{G,X^I})^{\mathrm{an}} \quad \text{and} \quad \mathrm{L}^+ G^{\mathrm{an}}_{X^I} \times_{X^{\mathrm{an}\,I}} \mathrm{Gr}^{\mathrm{an}}_{G,X^I} \to \mathrm{Gr}^{\mathrm{an}}_{G,X^I},$$

- *a stratified action over* $\mathrm{Ran}(X)^{\mathrm{an}}$

$$\mathrm{L}^+ G^{\mathrm{an}}_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)^{\mathrm{an}}} \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)} \to \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)}.$$

Moreover, the above analytifications are related by the following formulas:

$$\mathrm{Ran}(X)^{\mathrm{an}} = \operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} (X^{\mathrm{an}})^I$$

$$\mathrm{Gr}^{\mathrm{an}}_G = \operatorname*{colim}_{N \geq 0} (\mathrm{Gr}^{(N)}_G)^{\mathrm{an}}, \quad \mathrm{Gr}^{\mathrm{an}}_{G,X^I} = \operatorname*{colim}_{N \geq 0} (\mathrm{Gr}^{(N)}_{G,X^I})^{\mathrm{an}}, \quad \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)} = \operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \mathrm{Gr}^{\mathrm{an}}_{G,X^I}, \tag{4.1}$$
$$\mathrm{L}^+ G^{\mathrm{an}} = \lim_{m \geq 0} \mathrm{L}^m G^{\mathrm{an}}, \quad \mathrm{L}^+ G^{\mathrm{an}}_{X^I} = \lim_{m \geq 0} \mathrm{L}^m G^{\mathrm{an}}_{X^I}, \quad \mathrm{L}^+ G^{\mathrm{an}}_{\mathrm{Ran}(X)} = \operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \mathrm{L}^+ G^{\mathrm{an}}_{X^I}.$$

Proof. By Remark 3.29, we have

$$\mathrm{L}^m G_{X^I} \in \mathrm{Grp}\left(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}\ /X^I}\right), \quad \mathrm{L}^+ G_{X^I} \in \mathrm{ProGrp}\left(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}\ /X^I}\right).$$

Note that restricting to the fibers $x \in X(R)$, we get an induced stratifications on $\mathrm{L}^m G$ and $\mathrm{L}^+ G$, which coincides with the trivial ones. By Theorem 2.9, we can take their analytification which will have the form of (4.1) because $(-)^{\mathrm{an}}$ on $\mathrm{ProGrp}(\mathrm{SchStr}^{\mathrm{lft}}_{\mathbb{C}\ /X^I})$ preserves small limits.

Similarly, by Recall 3.5, we have that $\mathrm{Gr}^{(N)}_G \in \mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}}$, $\mathrm{Gr}_G \in \mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})$, and by Recall 3.15, we have that $\mathrm{Gr}^{(N)}_{G,X^I} \in \mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}\ /X^I}$, $\mathrm{Gr}_{G,X^I} \in \mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/X^I}$. By Remark 3.25, we have that $\mathrm{Ran}(X), \mathrm{Gr}_{G,\mathrm{Ran}(X)} \in \mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})$. By Theorem 2.9, they also have the description of (4.1) as colimits in StrTop because $(-)^{\mathrm{an}}_{\mathrm{PShStr}}$ on $\mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})$ preserves colimits. Now, we have stratified actions on Gr_G and (over X^I) on $\mathrm{Gr}^{(N)}_{G,X^I}$'s and on Gr_{G,X^I} (by Proposition 3.38). Since $(-)^{\mathrm{an}}_{\mathrm{Str}}$ preserves finite limits we have

$$(\mathrm{L}^m G \times \mathrm{Gr}^{(N)}_G)^{\mathrm{an}}_{\mathrm{Str}} = \mathrm{L}^m G^{\mathrm{an}} \times (\mathrm{Gr}^{(N)}_G)^{\mathrm{an}}, \text{ and } (\mathrm{L}^m G_{X^I} \times_{X^I} \mathrm{Gr}^{(N)}_{G,X^I})^{\mathrm{an}}_{\mathrm{Str}} = \mathrm{L}^m G^{\mathrm{an}}_{X^I} \times_{X^{\mathrm{an}\,I}} (\mathrm{Gr}^{(N)}_{G,X^I})^{\mathrm{an}},$$

hence we get stratified actions on the analytic side at the truncated level. Notice that the underlying poset is the same one for all $\mathrm{L}^m G$'s, and for all $\mathrm{L}^m G_{X^I}$'s as well (respectively $\{*\}$ for $\mathrm{L}^m G$'s and the incidence stratification poset of X^I for $\mathrm{L}^m G_{X^I}$'s), by the last part of Remark 2.2, we have that

$$\lim_m (\mathrm{L}^m G^{\mathrm{an}} \times (\mathrm{Gr}^{(N)}_G)^{\mathrm{an}}) = (\lim_m \mathrm{L}^m G^{\mathrm{an}}) \times (\mathrm{Gr}^{(N)}_G)^{\mathrm{an}} = \mathrm{L}^+ G^{\mathrm{an}} \times (\mathrm{Gr}^{(N)}_G)^{\mathrm{an}}$$

and similar for $\mathrm{L}^+ G^{\mathrm{an}}_{X^I} \times_{X^{\mathrm{an}\,I}} (\mathrm{Gr}^{(N)}_{G,X^I})^{\mathrm{an}}$.

Building on these data, we now construct the other group structures/actions. Notice now that after one proves the statement about the the existence of a stratified action of $\mathrm{L}^+ G^{\mathrm{an}}_{\mathrm{Ran}(X)}$ on $\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)}$, then one deduces existence of the stratified actions on $\mathrm{Gr}^{\mathrm{an}}_G$ and on $\mathrm{Gr}^{\mathrm{an}}_{G,X^I}$ over X^I by pullback.

Therefore, let us focus on $\mathrm{L}^+ G^{\mathrm{an}}_{\mathrm{Ran}(X)}$ and $\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)}$: we have seen in Proposition 3.32 that $\mathrm{L}^+ G_{\mathrm{Ran}(X)}$ is a group object in $\mathrm{Grp}(\mathrm{PSh}^{\mathrm{small}}(\mathrm{ProStrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/\mathrm{Ran}(X)})$. The analytification functor at the level of

$$(-)^{\mathrm{an}} : \mathrm{PSh}^{\mathrm{small}}(\mathrm{ProStrSch}^{\mathrm{lft}}_{\mathbb{C}})_{/\mathrm{Ran}(X)} \to \mathrm{StrTop}_{/\mathrm{Ran}(X)^{\mathrm{an}}}$$

may not preserve finite limits (or even be Cartesian lax-monoidal), and hence it does not automatically enhance to a functor between categories of group-objects. Moreover, although by (3.10) we have that

$$\mathrm{L}^+G_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)} \mathrm{L}^+G_{\mathrm{Ran}(X)} \simeq \operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}} \left(\mathrm{L}^+G_{X^I} \times_{X^I} \mathrm{L}^+G_{X^I}\right),$$

the same argument used in the proof of this formula cannot be applied in StrTop, since this latter is not a topos (hence, universality of colimits fails). For this reason, we need to adopt a different strategy in order to recover a group structure for $\mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(X)}$ and an action of it onto $\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)}$, relative over $\mathrm{Ran}(X)^{\mathrm{an}}$.

Define for $n \geq 1$ the stratified presheaf $\mathrm{Ran}_{\leq n}(X)$ which is the colimit

$$\mathrm{Ran}_{\leq n}(X) = \operatorname*{colim}^{\mathrm{PSh}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}, |I|\leq n} X^I.$$

Its functor of points parametrizes

$$\{\underline{x} \subset X(R) \mid |\underline{x}| \leq n\}.$$

Moreover, we have that $\mathrm{Ran}(X) \simeq \operatorname{colim}_{n\geq 1} \mathrm{Ran}_{\leq n}(X)$ in $\mathrm{PSh}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})$ and (taking the analytifications

$$\mathrm{Ran}_{\leq n}(X)^{\mathrm{an}} \simeq \operatorname*{colim}^{\mathrm{StrTop}}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}, |I|\leq n} (X^{\mathrm{an}})^I, \quad \mathrm{Ran}(X)^{\mathrm{an}} \simeq \operatorname*{colim}^{\mathrm{StrTop}}_{n\geq 1} \mathrm{Ran}_{\leq n}(X)^{\mathrm{an}}.$$

We can similarly set

$$\mathrm{Gr}_{G,\mathrm{Ran}_{\leq n}(X)} = \mathrm{Gr}_{G,\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)} \mathrm{Ran}_{\leq n}(X),$$

and denote by $\mathrm{Gr}^{(N)}_{G,\mathrm{Ran}_{\leq n}(X)}$ the image of $\mathrm{Gr}^{(N)}_{G,X^n}$ under the functor $\mathrm{Gr}_{G,X^n} \to \mathrm{Gr}_{G,\mathrm{Ran}_{\leq n}(X)}$. Then we obtain

$$\begin{gathered}
\mathrm{Gr}^{(N)}_{G,\mathrm{Ran}_{\leq n}(X)} \simeq \operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}, |I|\leq n} \mathrm{Gr}^{(N)}_{G,X^I}, \quad (\mathrm{Gr}^{(N)}_{G,\mathrm{Ran}_{\leq n}(X)})^{\mathrm{an}} \simeq \operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}, |I|\leq n} (\mathrm{Gr}^{(N)}_{G,X^I})^{\mathrm{an}} \\
\mathrm{Gr}_{G,\mathrm{Ran}(X)} \simeq \operatorname*{colim}_{n\geq 1, N\geq 0} \mathrm{Gr}^{(N)}_{G,\mathrm{Ran}_{\leq n}(X)}, \quad \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)} \simeq \operatorname*{colim}_{n\geq 1, N\geq 0} (\mathrm{Gr}^{(N)}_{G,\mathrm{Ran}_{\leq n}(X)})^{\mathrm{an}}.
\end{gathered} \tag{4.2}$$

Similar notations and isomorphisms hold for the arc group case:

$$\begin{gathered}
\mathrm{L}^+G_{\mathrm{Ran}_{\leq n}(X)} \simeq \operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}, |I|\leq n} \mathrm{L}^+G_{X^I}, \quad \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)} \simeq \operatorname*{colim}_{I\in \mathrm{Fin}^{\mathrm{op}}_{\geq 1,\mathrm{surj}}, |I|\leq n} \mathrm{L}^+G^{\mathrm{an}}_{X^I} \\
\mathrm{L}^+G_{\mathrm{Ran}(X)} \simeq \operatorname*{colim}_{n\geq 1} \mathrm{L}^+G_{\mathrm{Ran}_{\leq n}(X)}, \quad \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(X)} \simeq \operatorname*{colim}_{n\geq 1} \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)}.
\end{gathered} \tag{4.3}$$

The advantage of this reformulation is that two-fold, as we are about to see: on one hand, $\mathrm{Ran}_{\leq n}(X)^{\mathrm{an}}$, $\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}_{\leq n}(X)}$, $\mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)}$ are locally compact Hausdorff spaces, and filtered colimits of locally compact Hausdorff spaces commute with finite limits; on the other hand, these spaces can be presented as finite colimits enjoying a special property, i.e. they are the target of continuous closed surjections.

Let us focus on the latter first. By [CL21, Lemma 2.3] $(X^{\mathrm{an}})^n \to \mathrm{Ran}_{\leq n}(X)^{\mathrm{an}}$ is a closed continuous surjections. Now, such maps are closed under finite limits: therefore we get that the maps $(\mathrm{Gr}^{(N)}_{G,X^n})^{\mathrm{an}} \to (\mathrm{Gr}^{(N)}_{G,\mathrm{Ran}_{\leq n}(X)})^{\mathrm{an}}, \mathrm{L}^+G^{\mathrm{an}}_{X^n} \to \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)}$ are continuous closed surjections as well because pullbacks of the first one. In particular, they are all closed topological quotients.

Now, the sources of these maps are locally compact Hausdorff spaces, since they arise as analytifications of quasiprojective complex schemes. Hence, $\mathrm{Ran}_{\leq n}(X)^{\mathrm{an}}, \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}_{\leq n}(X)}, \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)}$ are locally compact Hausdorff spaces as well: the Hausdorffness comes from [Mat20] (cfr. also [Eng89, Theorem 4.4.15]) and the local compactnes from [CHT03, Proposition 4.1]

We can thus apply [Har15] and obtain the following formulas

$$\begin{aligned} \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)^{\mathrm{an}}} \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(X)} &\simeq \operatorname*{colim}_{n\geq 1} \left(\mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)} \times_{\mathrm{Ran}_{\leq n}(X)^{\mathrm{an}}} \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)}\right) \\ \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)^{\mathrm{an}}} \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)} &\simeq \operatorname*{colim}_{n\geq 1, N\geq 0} \left(\mathrm{L}^+G^{\mathrm{an}}_{G,\mathrm{Ran}_{\leq n}(X)} \times_{\mathrm{Ran}_{\leq n}(X)^{\mathrm{an}}} (\mathrm{Gr}^{(N)}_{G,\mathrm{Ran}_{\leq n}(X)})^{\mathrm{an}}\right). \end{aligned} \tag{4.4}$$

Now, and they are topological quotients, the maps

$$\begin{aligned} u_n : \mathrm{L}^+G^{\mathrm{an}}_{X^n} \times_{(X^{\mathrm{an}})^n} (\mathrm{Gr}^{(N)}_{G,X^n})^{\mathrm{an}} &\to \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)} \times_{\mathrm{Ran}_{\leq n}(X)^{\mathrm{an}}} (\mathrm{Gr}^{(N)}_{G,\mathrm{Ran}_{\leq n}(X)})^{\mathrm{an}} \\ u_n^+ : \mathrm{L}^+G^{\mathrm{an}}_{X^n} \times_{(X^{\mathrm{an}})^n} \mathrm{L}^+G^{\mathrm{an}}_{X^n} &\to \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)} \times_{\mathrm{Ran}_{\leq n}(X)^{\mathrm{an}}} \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)} \end{aligned}$$

are topological quotients (by the relation $(x_1, \dots, x_n, g) \sim (x'_1, \dots, x'_n, g') \iff \{x_1, \dots, x_n\} = \{x'_1, \dots, x'_n\}, g = g'$). Hence, the map

$$\mathrm{L}^+G^{\mathrm{an}}_{X^n} \times_{(X^{\mathrm{an}})^n} \mathrm{L}^+G^{\mathrm{an}}_{X^n} \to \mathrm{L}^+G^{\mathrm{an}}_{X^n} \to \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)}$$

induced by the multiplication of $\mathrm{L}^+G_{X^n}$ factors through u_n^+, yielding a well-defined continuous stratified relative group law

$$\mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)} \times_{\mathrm{Ran}_{\leq n}(X)^{\mathrm{an}}} \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)} \to \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}_{\leq n}(X)}.$$

By (4.4), this reassembles to a well-defined continuous stratified relative group law

$$\mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)^{\mathrm{an}}} \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(X)} \to \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(X)}.$$

In the same way, one produces a continuous stratified relative group action

$$\mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(X)} \times_{\mathrm{Ran}(X)^{\mathrm{an}}} \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)} \to \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)}.$$

□

Definition 4.2. Consider the analytified forgetful maps

$$(\mathrm{Gr}^{(N)}_{G,X^I})^{\mathrm{an}} \to (X^{\mathrm{an}})^I, \quad \mathrm{Gr}^{\mathrm{an}}_{G,X^I} \to (X^{\mathrm{an}})^I.$$

Given an open subset $D \subseteq X^{\mathrm{an}}$ we then define $\mathrm{Gr}^{(N)}_{G,D^I}$ as the fiber product (taken in StrTop)

$$\begin{tikzcd} \mathrm{Gr}^{(N)}_{G,D^I} \arrow[r, hook] \arrow[d] & (\mathrm{Gr}^{(N)}_{G,X^I})^{\mathrm{an}} \arrow[d] \\ D^I \arrow[r, hook] & (X^{\mathrm{an}})^I \end{tikzcd}$$

and Gr_{G,D^I} as the fiber product (again in StrTop)

$$\begin{tikzcd} \mathrm{Gr}_{G,D^I} \arrow[r, hook] \arrow[d] & \mathrm{Gr}^{\mathrm{an}}_{G,X^I} \arrow[d] \\ D^I \arrow[r, hook] & (X^{\mathrm{an}})^I. \end{tikzcd}$$

We define analogously $\mathrm{L}^mG_{D^I}, \mathrm{L}^+G_{D^I}$ as fiber products in StrTop.

In order to define $\mathrm{Gr}_{G,\mathrm{Ran}(D)}$, we need first to recall what we mean by the *Ran space* of a topological manifold.

Definition 4.3. Given M, denote by $\mathrm{Ran}(M)$ the colimit $\operatorname*{colim}_{I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}} M^I$, where the M^I's carry the incidence stratification and the colimit is done in StrTop). We call it the *Ran space* of M.

Remark 4.4. There are several versions of the Ran space: ours is the one with the colimit topology, but the so-called metric topology is also relevant, see [CL21].

Remark 4.5. The association $M \mapsto \mathrm{Ran}(M)$ is functorial, by functoriality of the colimit. Moreover, if $M' \to M$ is an open immersion, so is $\mathrm{Ran}(M') \to \mathrm{Ran}(M)$ by definition of colimit topology. Therefore, given an open subset $D \subset X^{\mathrm{an}}$, we get an open immersion

$$\mathrm{Ran}(D) \hookrightarrow \mathrm{Ran}(X^{\mathrm{an}}) = \mathrm{Ran}(X)^{\mathrm{an}}.$$

In particular, since $\mathrm{Ran}(X)^{\mathrm{an}}$ is a stratified space, $\mathrm{Ran}(D)$ becomes a stratified space via restriction of the strata.

Definition 4.6. Consider the analytified forgetful map $\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)} \to \mathrm{Ran}(X)^{\mathrm{an}}$. Let D be an open subset of X^{an}: define $\mathrm{Gr}_{G,\mathrm{Ran}(D)}$ to be the fibered product (taken in StrTop)

$$\begin{tikzcd} \mathrm{Gr}_{G,\mathrm{Ran}(D)} \arrow[r, hook] \arrow[d] & \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)} \arrow[d] \\ \mathrm{Ran}(D) \arrow[r] & \mathrm{Ran}(X)^{\mathrm{an}}. \end{tikzcd} \tag{4.5}$$

4.2 Lifting isotopies

Most of the proof of the main result of the paper, Corollary 4.14, is based on the following two lemmas.

Lemma 4.7. *Let R be a $\mathbb{C}$-algebra, and let $f : X_R \to X_R$ be an R-linear automorphism. This induces an automorphism of ind-R-schemes $\Phi_f : (\mathrm{Gr}_{G,X^I})_R \to (\mathrm{Gr}_{G,X^I})_R$. As a consequence, there is a morphism of presheaves*

$$\Phi : \underline{\mathrm{Aut}}_{\mathbb{C}}(X) \to \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{Gr}_{G,X^I}), \quad f \mapsto \Phi_f. \tag{4.6}$$

Proof. For any $\mathbb{C}$-algebra A, we want to exhibit a bijection of the A-points

$$\Phi_{f,A} : \left(\mathrm{Gr}_{G,X^I} \times_{\mathbb{C}} \mathrm{Spec}\, R\right)(A) \to \left(\mathrm{Gr}_{G,X^I} \times_{\mathbb{C}} \mathrm{Spec}\, R\right)(A)$$

natural in A. Let $a : \mathrm{Spec}\, A \to \mathrm{Gr}_{G,X^I} \times_{\mathbb{C}} \mathrm{Spec}\, R$ be an A-point, corresponding to

$x_I \in X^I(A),\ \mathcal{F} \in \mathrm{Bun}_G(X_A),\ \alpha : \mathcal{F}|_{X_A \setminus \Gamma_{x_I}} \xrightarrow{\sim} \mathcal{T}_{G,X_A \setminus \Gamma_{x_I}}$, and a $\mathbb{C}$-linear map $\tau : \mathrm{Spec}\, A \to \mathrm{Spec}\, R$.

Let f_A be the automorphism of X_A obtained by pullback of f along τ. Let y_I be the composition

$$\mathrm{Spec}\, A \xhookrightarrow{x_I \times \mathrm{id}_A} X_A^I \xrightarrow{(f_A^{-1})^I} X_A^I \xrightarrow{\mathrm{pr}_{X^I}} X^I.$$

Define

$$\Phi_{f,A}(x_I, \mathcal{F}, \alpha, \tau) = (y_I, f_A^* \mathcal{F}, f_A^* \alpha, \tau)$$

(well-defined because $f_A^*(\mathcal{F}|_{X_A \setminus \Gamma_{x_I}}) \simeq (f_A^* \mathcal{F})|_{X_A \setminus \Gamma_{y_I}}$ and $f_A^* \mathcal{T}_{G,X_A} \simeq \mathcal{T}_{G,X_A}$). Since the formation of f_A is natural in A, so is $\Phi_{f,A}$. $\square$

Lemma 4.8. *For any N, the morphism (4.6) sends an automorphism $f : X \to X$ to a $\mathbb{C}$-automorphism of the stratified $\mathbb{C}$-scheme $\mathrm{Gr}^{(N)}_{G,X^I}$. In particular Φ_f upgrades to an element of $\mathrm{Aut}_{\mathrm{PSh}^{\mathrm{small}}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})}(\mathrm{Gr}_{G,X^I})$.*

Proof. Consider a stratum $\mathrm{Gr}_{G,X^{\phi},\underline{\nu}}$ of $\mathrm{Gr}^{(N)}_{G,X^I}$. Since the map $f^I : X^I \to X^I$ respects the incidence stratification on X^I, then Φ_f sends $\left(\prod_{j=1}^{|J|} \mathrm{Gr}_{G,X}\right)^{\circ}$ to itself: by factorization it is therefore enough to check that $\mathrm{Gr}_{G,X,\nu}$'s are preserved by Φ_f (so $I = \{*\}$).

In order to do this, we use the isomorphism $\widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]} \mathrm{Gr}_{G,\nu} \xrightarrow{\sim} \mathrm{Gr}_{G,X,\nu}$ from Proposition 3.13. Pick an A-point

$$[(x, \eta, \widetilde{\mathcal{F}}, \widetilde{\alpha})] : \mathrm{Spec}\, A \to (\widehat{X} \times \mathrm{Gr}_{G,\nu})/\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]$$

and let $(x, \mathcal{F}, \alpha)$ be an A-point in $\mathrm{Gr}_{G,X,\nu}$ such that

$$\eta^* i^*_{\widehat{x}} \mathcal{F} \simeq \widetilde{\mathcal{F}}, \quad (i_{\widehat{x}} \circ \eta)|^*_{\mathrm{Spec}\, R(\!(t)\!)} \alpha \simeq \widetilde{\alpha}.$$

(this is uniquely determined up to isomorphism). Now $\Phi_{f,A}(x, \mathcal{F}, \alpha) = (f^{-1}x, f^*_A \mathcal{F}, f^*_A \alpha)$, which means

$$(f_A^{-1} \circ i_{\widehat{x}} \circ \eta)^* f^*_A \mathcal{F} \simeq \widetilde{\mathcal{F}}, \quad (f_A^{-1} \circ i_{\widehat{x}} \circ \eta)|^*_{\mathrm{Spec}\, R(\!(t)\!)} f^*_A|_{\mathrm{Spec}\, R(\!(t)\!)} \alpha \simeq \widetilde{\alpha}. \tag{4.7}$$

Consider the cartesian diagram

$$\begin{tikzcd} \mathrm{Spec}\, A[\![t]\!] \arrow[r, "\widehat{f}^{-1}_{A,x} \circ \eta"] \arrow[d, equal] & \underline{\mathrm{Spec}}_{X_A}(\widehat{\mathcal{O}}_{\Gamma_{f^{-1}x}}) \arrow[r, "i_{\widehat{f^{-1}x}}"] \arrow[d, "\widehat{f}_{A,x}"] & X_A \arrow[d, "f_A"] \\ \mathrm{Spec}\, A[\![t]\!] \arrow[r, "\eta"] & \underline{\mathrm{Spec}}_{X_A}(\widehat{\mathcal{O}}_{\Gamma_x}) \arrow[r, "i_{\widehat{x}}"] & X_A. \end{tikzcd} \tag{4.8}$$

Then we can rewrite (4.7) as

$$(\widehat{f}^{-1}_{A,x} \circ \eta)^* i^*_{\widehat{f^{-1}x}}(f^*_A \mathcal{F}) \simeq \widetilde{\mathcal{F}}, \quad (\widehat{f}^{-1}_{A,x} \circ \eta|_{\mathrm{Spec}\, R(\!(t)\!)})^* i^*_{\widehat{f^{-1}x}}(f^*_A \alpha) \simeq \widetilde{\alpha}.$$

Therefore $\Phi_{f,A}(x, \mathcal{F}, \alpha)$ corresponds to

$$(\mathrm{pr}_X f_A^{-1} x, \widehat{f}^{-1}_{A,x} \circ \eta, \widetilde{\mathcal{F}}, \widetilde{\alpha}) : \mathrm{Spec}\, A \to \widehat{X} \times \mathrm{Gr}_{G,\nu}/\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]. \tag{4.9}$$

In particular, passing to the sheafification, this implies that Φ_f only modifies the first components of the twisted product and therefore it preserves the stratification. By a similar argument, one can see that the map sends $\mathrm{Gr}^{(N)}_{G,X^I}$ to itself. $\square$

Definition 4.9. Let Y be a presheaf in $\mathrm{PSh}(\mathrm{StrSch}^{\mathrm{lft}}_{\mathbb{C}})$. An *algebraic isotopy* of Y is a morphism in $\mathrm{PSh}(\mathrm{Aff}_{\mathbb{C}})$

$$F : U \to \underline{\mathrm{Aut}}_{\mathbb{C}}(Y),$$

where U is an open of $\mathbb{A}^1_{\mathbb{C}}$ such that $[0, 1] \subset U^{\mathrm{an}}$.

Remark 4.10. Given an algebraic isotopy of X, Lemma 4.7 tells us that we get an algebraic isotopy

$$\Phi_U = \Phi \circ F : U \to \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{Gr}_{G,X^I}).$$

Let us consider U as a stratified scheme with the trivial stratification. Composing with the evaluation

$$\mathrm{ev} : \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{Gr}_{G,X^I}) \times_{\mathbb{C}} \mathrm{Gr}_{G,X^I} \to \mathrm{Gr}_{G,X^I}, \qquad (f, x) \mapsto f(x)$$

by Lemma 4.8 we get a map of stratified ind-schemes

$$\mathrm{ev} \circ (\Phi_U \times \mathrm{id}_{\mathrm{Gr}_{G,X^I}}) : U \times_{\mathbb{C}} \mathrm{Gr}_{G,X^I} \to \mathrm{Gr}_{G,X^I}. \tag{4.10}$$

Via the analytification functor we then get a continuous map in StrTop

$$\Psi_U : U^{\mathrm{an}} \times \mathrm{Gr}^{\mathrm{an}}_{G,X^I} \to \mathrm{Gr}^{\mathrm{an}}_{G,X^I}, \tag{4.11}$$

which from a set-theoretical point of view coincides with $\mathrm{ev} \circ (\Phi_U \times \mathrm{id}_{\mathrm{Gr}_{G,X^I}})_{\mathbb{C}}$ (namely, $\mathrm{ev} \circ (\Phi_U \times \mathrm{id}_{\mathrm{Gr}_{G,X^I}})$ at the level of $\mathbb{C}$-points). Therefore, for every $z \in U^{\mathrm{an}} = U(\mathbb{C})$, the map $\Psi_U(z,-) = \Phi_U^{\mathrm{an}}(z)(-)$ is a map in $\mathrm{Aut}_{\mathrm{StrTop}}(\mathrm{Gr}^{\mathrm{an}}_{G,X^I})$. In particular, we get a topological isotopy

$$\Psi_{[0,1]} = \Psi_U|_{[0,1]} : [0,1] \times \mathrm{Gr}^{\mathrm{an}}_{G,X^I} \to \mathrm{Gr}^{\mathrm{an}}_{G,X^I}. \tag{4.12}$$

Definition 4.11. Let $f, g : (Y, s_Y) \to (W, s_W)$ be two maps of stratified topological spaces. Let $\widetilde{s}_Y$ be the stratification of $[0,1] \times Y$ induced by the projection $[0,1] \times Y \to Y$. A *stratified homotopy* between f and g is a stratified map

$$H : ([0,1] \times Y, \widetilde{s}_Y) \to (W, s_W)$$

such that $H(0,-) = f, H(1,-) = g$.

Definition 4.12. A *stratified homotopy equivalence* of stratified topological spaces is then a stratified map $f : (Y, s_Y) \to (W, s_W)$ such that there exist a stratified map $g : (W, s_W) \to (Y, s_Y)$ and stratified homotopies $gf \sim \mathrm{id}_{(Y,s_Y)}, fg \sim \mathrm{id}_{(W,s_W)}$.

Theorem 4.13. *Let $F : U \to \underline{\mathrm{Aut}}_{\mathbb{C}}(X)$ be an algebraic isotopy. Consider two open $D' \subset D \subset X^{\mathrm{an}}$ and suppose that*

1. *for every $t \in [0,1] \subset U^{\mathrm{an}}$ we have $F_t^{\mathrm{an}}(D') \subset D'$ and $F_t^{\mathrm{an}}(D) \subset D$,*
2. *$F_0^{\mathrm{an}}|_D = \mathrm{id}_D$ and $F_1^{\mathrm{an}}(D) = D'$.*

Then the open inclusions

$$i^{(N)} : \mathrm{Gr}^{(N)}_{G,D'^I} \hookrightarrow \mathrm{Gr}^{(N)}_{G,D^I}, \quad \text{and} \quad i : \mathrm{Gr}_{G,D'^I} \hookrightarrow \mathrm{Gr}_{G,D^I},$$

are stratified homotopy equivalences, and the homotopies involved can be taken to be isotopies.

Proof. Consider the map

$$\Psi_{[0,1]} : [0,1] \times \mathrm{Gr}^{\mathrm{an}}_{G,X^I} \to \mathrm{Gr}^{\mathrm{an}}_{G,X^I}$$

from (4.12). By condition 1, for any $t \in [0,1]$ the image of $\Psi_t|_{\mathrm{Gr}_{G,D^I}} = \Psi_{[0,1]}(t,-)|_{\mathrm{Gr}_{G,D^I}}$ lies all in Gr_{G,D^I} (the same holds for D'). By condition 2,

$$\Psi_0|_{\mathrm{Gr}_{G,D^I}} = \mathrm{id}_{\mathrm{Gr}_{G,D^I}}, \quad \text{and} \quad \Psi_1|_{\mathrm{Gr}_{G,D^I}}(\mathrm{Gr}_{G,D^I}) \subset \mathrm{Gr}_{G,D'^I}.$$

We claim that $\Psi_1|_{\mathrm{Gr}_{G,D^I}} : \mathrm{Gr}_{G,D^I} \to \mathrm{Gr}_{G,D'^I}$ is a stratified homotopy inverse to the inclusion $i : \mathrm{Gr}_{G,D'^I} \hookrightarrow \mathrm{Gr}_{G,D^I}$. Consider first $i \circ \Psi_1|_{\mathrm{Gr}_{G,D^I}}$: then

$$\Psi_{[0,1]}|_{\mathrm{Gr}_{G,D^I}} : [0,1] \times \mathrm{Gr}_{G,D^I} \to \mathrm{Gr}_{G,D^I}$$

gives a stratified isotopy between $\mathrm{id}_{\mathrm{Gr}_{G,D^I}}$ and $i \circ \Psi_1^{\mathrm{an}}|_{\mathrm{Gr}_{G,D^I}}$.

Consider now $\Psi_1|_{\mathrm{Gr}_{G,D^I}} \circ i = \Psi_1|_{\mathrm{Gr}_{G,D'^I}}$: by condition 1 for any t the image $\Psi_t|_{\mathrm{Gr}_{G,D'^I}}$ is contained in Gr_{G,D'^I}). Then

$$\Psi_{[0,1]}|_{\mathrm{Gr}_{G,D'^I}} : [0,1] \times \mathrm{Gr}_{G,D'^I} \to \mathrm{Gr}_{G,D'^I}$$

gives a stratified isotopy between $\mathrm{id}_{\mathrm{Gr}_{G,D'^I}}$ and $\Psi_1|_{\mathrm{Gr}_{G,D^I}} \circ i$.

The case of $i^{(N)} : \mathrm{Gr}^{(N)}_{G,D'^I} \hookrightarrow \mathrm{Gr}^{(N)}_{G,D^I}$ is analogous (see Lemma 4.8). $\square$

Corollary 4.14. *Let $z_0, z_0' \in \mathbb{C}$, and $r > r' \in \mathbb{R}_{>0}$ such that $B(z_0', r') \subset B(z_0, r) \subset \mathbb{C}$. Denote by D' the ball $B(z_0', r')$, and by D the ball $B(z_0, r)$. The induced open embeddings*

$$i^{(N)} : \mathrm{Gr}^{(N)}_{G,D'^I} \hookrightarrow \mathrm{Gr}^{(N)}_{G,D^I}, \quad \text{and} \quad i : \mathrm{Gr}_{G,D'^I} \hookrightarrow \mathrm{Gr}_{G,D^I}$$

are stratified homotopy equivalences, and the homotopies involved can be taken to be isotopies.

Proof. Consider the map

$$F : \mathbb{A}^1_{\mathbb{C}} \to \underline{\mathrm{End}}_{\mathbb{C}}(\mathbb{A}^1_{\mathbb{C}}) = \underline{\mathrm{End}}_{\mathbb{C}}(\mathrm{Spec}\,\mathbb{C}[z]), \qquad t \in R \mapsto F_R(t) : z \mapsto z\Big(\frac{r'}{r}t + (1-t)\Big) + t\Big(z_0' - \frac{r'}{r}z_0\Big).$$

Notice that $F_R(t)$ is an an automorphism of $\mathbb{A}^1_R$ if and only if the scaling factor $\lambda(t) = \frac{r'}{r}t + (1-t)$ is in $R^\times = \mathbb{G}_{m,\mathbb{C}}(R)$. This happens if and only if $\lambda(t)$ belongs to the open $U \subseteq \mathbb{A}^1_{\mathbb{C}}$

$$\begin{tikzcd} U \arrow[r, dashed] \arrow[d, hook, dashed] & \mathbb{G}_{m,\mathbb{C}} \arrow[d, hook] \\ \mathbb{A}^1_{\mathbb{C}} \arrow[r, "\lambda"] & \mathbb{A}^1_{\mathbb{C}}. \end{tikzcd}$$

If $t \in \mathbb{C}$, then $\lambda(t) \notin \mathbb{C}^\times$ if and only if $t = \frac{r}{r-r'}$: since $r > r'$, then $[0,1] \subset U^{\mathrm{an}}$. Then $F|_U$ is an algebraic isotopy in the sense of Definition 4.9 and it satisfies the hypotheses of Theorem 4.13. $\square$

Corollary 4.15. *Let $D' \subset D \subset \mathbb{C}$ be as in Corollary 4.14. The induced open embedding*

$$i : \mathrm{Gr}_{G,\mathrm{Ran}(D')} \hookrightarrow \mathrm{Gr}_{G,\mathrm{Ran}(D)}$$

is a stratified homotopy equivalence, and the homotopies involved can be taken to be isotopies.

Proof. The map

$$\underline{\mathrm{Aut}}_{\mathbb{C}}(X) \to \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{Gr}_{G,X^I})$$

in Lemma 4.7 is natural in $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$. Therefore, it upgrades to a morphism of presheaves

$$\Psi^{\mathrm{Ran}} : \underline{\mathrm{Aut}}_{\mathbb{C}}(X) \to \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{Gr}_{G,\mathrm{Ran}(X)}).$$

By arguing as in Remark 4.10, given any algebraic isotopy $U \to \underline{\mathrm{Aut}}_C(X)$, we obtain a stratified map

$$U^{\mathrm{an}} \times \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)} \to \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)}$$

and hence a stratified homotopy iotopy

$$\Psi^{\mathrm{Ran}}_{[0,1]} : [0,1] \times \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)} \to \mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)}.$$

The analogues of Theorem 4.13 and Corollary 4.14 are deduced in the same way. $\square$

4.3 Equivariance

Remark 4.16. Note that there are morphisms of presheaves

$$\Phi^{\mathrm{L}^+G}: \underline{\mathrm{Aut}}_{\mathbb{C}}(X) \to \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{L}^+G_{X^I}), \quad f \mapsto \Phi_f^{\mathrm{L}^+G}: (x_I, g) \mapsto ((f^{-1})^I(x_I), \widehat{f}^*_{x_I} g)$$
$$\Phi^{\mathrm{L}^+G_{\mathrm{Ran}(X)}}: \underline{\mathrm{Aut}}_{\mathbb{C}}(X) \to \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{L}^+G_{\mathrm{Ran}(X)}), \quad f \mapsto \Phi_f^{\mathrm{L}^+G}: (\underline{x}, g) \mapsto (f^{-1}(\underline{x}), \widehat{f}^*_{\underline{x}} g).$$

(notation as in proof of Lemma 4.8) and for every $m \in \mathbb{N}$

$$\Phi^{\mathrm{L}^mG}: \underline{\mathrm{Aut}}_{\mathbb{C}}(X) \to \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{L}^mG_{X^I}), \quad f \mapsto \Phi_f^{\mathrm{L}^mG}: (x_I, g) \mapsto ((f^{-1})^I(x_I), f|^*_{\Gamma^m_{(f^{-1})^I(x_I)}} g)$$

(same arguments as in Lemma 4.7).

Therefore one can straightforwardly transfer the proof of Theorem 4.13, Corollary 4.14 and Corollary 4.15 to prove the following:

Proposition 4.17. *Let $D' \subset D \subset \mathbb{C}$ be as in Corollary 4.14. Let $N \in \mathbb{N}$ and $m \geq m_{N,I}$. Then the induced open embeddings*

$$i^m: \mathrm{L}^mG_{D'^I} \hookrightarrow \mathrm{L}^mG_{D^I}$$
$$i^+: \mathrm{L}^+G_{D'^I} \hookrightarrow \mathrm{L}^+G_{D^I}$$
$$i^+_{\mathrm{Ran}}: \mathrm{L}^+G_{\mathrm{Ran}(D')} \hookrightarrow \mathrm{L}^+G_{\mathrm{Ran}(D)}$$

are stratified homotopy equivalences, and the homotopies involved can be taken to be isotopies.

Notice that by their definition, the open embedding $i^+: \mathrm{L}^+G_{D'^I} \hookrightarrow \mathrm{L}^+G_{D^I}$ together with the open embedding $i: \mathrm{Gr}_{G,D'^I} \hookrightarrow \mathrm{Gr}_{G,D^I}$ fit in the commutative diagram

$$\begin{tikzcd}
\mathrm{L}^+G_{D'^I} \times_{D'^I} \mathrm{Gr}_{G,D'^I} \arrow[r, hook, "i^+\times i"] \arrow[d, "\alpha_{D'^I}"] & \mathrm{L}^+G_{D^I} \times_{D^I} \mathrm{Gr}_{G,D^I} \arrow[d, "\alpha_{D^I}"] \\
\mathrm{Gr}_{G,D'^I} \arrow[r, hook, "i"] & \mathrm{Gr}_{G,D^I}
\end{tikzcd}$$

where the vertical maps are the action maps. The same is true for the L^mG-version and for the Ran-version.

Furthermore, all the mentioned isotopies in Corollary 4.14 and Proposition 4.17 are compatible with the above diagram (and its variations), in the following sense.

Theorem 4.18. *Let D be a metric disk in $\mathbb{C}$ and let $I \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$. There exists a stratified map*

$$\Psi^{\mathrm{equiv}}_{[0,1]}: [0,1] \times \left(\mathrm{L}^+G^{\mathrm{an}}_{(\mathbb{A}^1_{\mathbb{C}})^I} \times_{\mathbb{C}^I} \mathrm{Gr}^{\mathrm{an}}_{G,(\mathbb{A}^1_{\mathbb{C}})^I}\right) \to \mathrm{L}^+G^{\mathrm{an}}_{(\mathbb{A}^1_{\mathbb{C}})^I} \times_{\mathbb{C}^I} \mathrm{Gr}^{\mathrm{an}}_{G,(\mathbb{A}^1_{\mathbb{C}})^I}$$

such that

1. *for any $t \in [0,1]$, Ψ^{equiv}_t is a closed embedding;*
2. *it commutes with the action of $\mathrm{L}^+G^{\mathrm{an}}_{(\mathbb{A}^1_{\mathbb{C}})^I}$ and with the stratified homotopy $\Psi_{[0,1]}: [0,1] \times \mathrm{Gr}^{\mathrm{an}}_{G,(\mathbb{A}^1_{\mathbb{C}})^I} \to \mathrm{Gr}^{\mathrm{an}}_{G,(\mathbb{A}^1_{\mathbb{C}})^I}$. More precisely*

$$\begin{tikzcd}
{[0,1]} \times \mathrm{L}^+G_{D^I} \times_{D^I} \mathrm{Gr}_{G,D^I} \arrow[r, "\Psi^{\mathrm{equiv}}_{[0,1]}|_{\mathrm{Gr}_{G,D^I}}"] \arrow[d, "\mathrm{id}_{[0,1]}\times \mathrm{act}_{D^I}"] & \mathrm{L}^+G_{D^I} \times_{D^I} \mathrm{Gr}_{G,D^I} \arrow[d, "\mathrm{act}_{D^I}"] \\
{[0,1]} \times \mathrm{Gr}_{G,D^I} \arrow[r, "\Psi_{[0,1]}|_{\mathrm{Gr}_{G,D^I}}"] & \mathrm{Gr}_{G,D^I},
\end{tikzcd}$$

   commutes.

In particular, if $D' \subset D$ are metric disks, $\Psi^{\mathrm{equiv}}_{[0,1]}|_{\mathrm{Gr}_{G,D^I}}$ provides a stratified isotopy between id *and $(i^+ \times i) \circ \Psi^{\mathrm{equiv}}_1|_{\mathrm{Gr}_{G,D^I}}$, and $\Psi^{\mathrm{equiv}}_{[0,1]}|_{\mathrm{Gr}_{G,D'^I}}$ provides a stratified isotopy between* id *and $\Psi^{\mathrm{equiv}}_1|_{\mathrm{Gr}_{G,D^I}} \circ (i^+ \times i)$.*

An analogous statement holds for $\mathrm{L}^m G^{\mathrm{an}}_{(\mathbb{A}^1_{\mathbb{C}})^I} \times_{(\mathbb{C})^I} (\mathrm{Gr}^{(N)}_{G,(\mathbb{A}^1_{\mathbb{C}})^I})^{\mathrm{an}}$ for any $N \in \mathbb{N}$ and $m \geq m_{N,I}$.

Proof. The automorphisms Φ an Φ^{L^+G} act on the X^I-component in the same way (compare definition of Φ^{L^+G} in Remark 4.16 with the one of Φ given in the proof of Lemma 4.7). Therefore they can be combined together in order to obtain

$$\Phi^{\mathrm{L}^+G} \times_{X^I} \Phi : \underline{\mathrm{Aut}}_{\mathbb{C}}(X) \to \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{L}^+G_{X^I} \times_{X^I} \mathrm{Gr}_{G,X^I}).$$

Similarly, for any $N \in \mathbb{N}, m \geq m_{N,I}$, we have

$$\Phi^{\mathrm{L}^m G} \times_{X^I} \Phi : \underline{\mathrm{Aut}}_{\mathbb{C}}(X) \to \underline{\mathrm{Aut}}_{\mathbb{C}}(\mathrm{L}^m G_{X^I} \times_{X^I} \mathrm{Gr}^{(N)}_{G,X^I}).$$

Let F and U be as in Corollary 4.14. In particular consider $F^{\mathrm{an}}|_{[0,1]}$, which makes sense since $[0,1] \subseteq U^{\mathrm{an}}$. Note now that $(\Phi^{\mathrm{L}^+G} \times_{X^I} \Phi)^{\mathrm{an}}_{[0,1]}$ is the same as $(\Phi^{\mathrm{L}^+G})^{\mathrm{an}}_{[0,1]} \times_{\mathbb{C}^I} \Phi^{\mathrm{an}}_{[0,1]}$ because the analytification functor preserve finite limits. Denote by id the identity $\mathrm{id}_{\mathrm{L}^+G^{\mathrm{an}}_{X^I} \times_{\mathbb{C}^I} \mathrm{Gr}^{\mathrm{an}}_{G,X^I}}$. Therefore one can define

$$\Psi^{\mathrm{equiv}}_{[0,1]} = \mathrm{ev} \circ \left(\left((\Phi^{\mathrm{L}^+G})^{\mathrm{an}}_{[0,1]} \times_{\mathbb{C}^I} \Phi^{\mathrm{an}}_{[0,1]}\right) \times \mathrm{id}\right) : [0,1] \times \left(\mathrm{L}^+G^{\mathrm{an}}_{X^I} \times_{\mathbb{C}^I} \mathrm{Gr}^{\mathrm{an}}_{G,X^I}\right) \to \mathrm{L}^+G^{\mathrm{an}}_{X^I} \times_{\mathbb{C}^I} \mathrm{Gr}^{\mathrm{an}}_{G,X^I}$$

(and its restriction to Gr_{G,D^I} and Gr_{G,D'^I} respectively). Note that by definition, Ψ^{equiv}_1 is a map between

$$\mathrm{L}^+G_{D^I} \times_{D^I} \mathrm{Gr}_{G,D^I} \to \mathrm{L}^+G^{\mathrm{an}}_{D'^I} \times_{D'^I} \mathrm{Gr}^{\mathrm{an}}_{G,D'^I}$$

and, by the same proof of Theorem 4.13 and Corollary 4.14, it gives a stratified homotopy inverse to $i^+ \times i$. Therefore it suffices to show that, for any $t \in [0,1]$, $\Psi^{\mathrm{equiv}}_t|_{\mathrm{Gr}_{X,D^I}}$ and $\Psi_t|_{\mathrm{Gr}_{X,D^I}}$ fit in a commutative diagram

$$\begin{tikzcd}
\mathrm{L}^+G_{D^I} \times_{D^I} \mathrm{Gr}_{G,D^I} \arrow[r, "\Psi^{\mathrm{equiv}}_t|_{\mathrm{Gr}_{X,D^I}}"] \arrow[d] & \mathrm{L}^+G_{D^I} \times_{D^I} \mathrm{Gr}_{G,D^I} \arrow[d] \\
\mathrm{Gr}_{G,D^I} \arrow[r, "\Psi_t|_{\mathrm{Gr}_{X,D^I}}"] & \mathrm{Gr}_{G,D^I},
\end{tikzcd}$$

where the vertical maps are the action maps. This, in turn, is implied by checking that for any $f \in \underline{\mathrm{Aut}}_{\mathbb{C}}(X)$ and each locally closed subschemes Gr_{G,X^ϕ} of Gr_{G,X^I}, the diagram

$$\begin{tikzcd}
\mathrm{L}^+G_{X^I}|_{X^\phi} \times_{X^\phi} \mathrm{Gr}_{G,X^\phi} \arrow[r, "(\Phi^{\mathrm{L}^+G}_f \times_{X^I} \Phi_f)|_{X^\phi}"] \arrow[d] & \mathrm{L}^+G_{X^I}|_{X^\phi} \times_{X^\phi} \mathrm{Gr}_{G,X^\phi} \arrow[d] \\
\mathrm{Gr}_{G,X^\phi} \arrow[r, "\Phi_f|_{X^\phi}"] & \mathrm{Gr}_{G,X^\phi},
\end{tikzcd}$$

is well-defined and commutes.

By the factorization property (3.5), it is enough to deal with the $I = *$ case. In order to do so, we use the description with formal coordinates, $\mathrm{Gr}^{\mathrm{loc}}_{G,X} = \widehat{X} \times^{\underline{\mathrm{Aut}}_{\mathbb{C}}\mathbb{C}[\![t]\!]} \mathrm{Gr}_G \simeq \mathrm{Gr}_{G,X}$.

At the level of $\mathrm{Gr}^{\mathrm{loc}}_{G,X}$, the map Φ_f sends

$$(x,\eta,\widetilde{\mathcal{F}},\widetilde{\alpha}) \mapsto (f^{-1}x, \widehat{f}_x^{-1}\circ\eta, \widetilde{\mathcal{F}},\widetilde{\alpha})$$

(see equation (4.9)). Given $(x,g)\in \mathrm{L}^+G_X$, we one hand we have

$$\begin{tikzcd}
(x,g),(x,\eta,\widetilde{\mathcal{F}},\widetilde{\alpha}) \arrow[d] & \\
(x,\eta,\widetilde{\mathcal{F}},\eta^*g|_{\widetilde{\Gamma}_x\setminus\Gamma_x}\circ(\eta^{-1})^*\widetilde{\alpha}) \arrow[r, "\Phi_f"] & (f^{-1}x,\widehat{f}_x^{-1}\circ\eta,\widetilde{\mathcal{F}},\eta^*g|_{\widetilde{\Gamma}_x\setminus\Gamma_x}\circ(\eta^{-1})^*\widetilde{\alpha})
\end{tikzcd}$$

and on the other hand

$$\begin{tikzcd}
(x,g),(x,\eta,\widetilde{\mathcal{F}},\widetilde{\alpha}) \arrow[r, "\Phi_f^{\mathrm{L}^+G}\times_X\Phi_f"] & (f^{-1}x,\widehat{f}_x^*g),(f^{-1}x,\widehat{f}_x^{-1}\circ\eta,\widetilde{\mathcal{F}},\widetilde{\alpha}) \arrow[d] \\
& (f^{-1}x,\widehat{f}_x^{-1}\circ\eta,\widetilde{\mathcal{F}},(\widehat{f}_x^{-1}\circ\eta)^*(\widehat{f}_x^*g)|_{\widetilde{\Gamma}_{f^{-1}x}\setminus\Gamma_{f^{-1}x}}\circ((\widehat{f}_x^{-1}\circ\eta)^{-1})^*\widetilde{\alpha}).
\end{tikzcd}$$

One concludes computing explicitly the last term:

$$\begin{aligned}
(\widehat{f}_x^{-1}\circ\eta)^*(\widehat{f}_x^*g)|_{\widetilde{\Gamma}_{f^{-1}x}\setminus\Gamma_{f^{-1}x}}\circ((\widehat{f}_x^{-1}\circ\eta)^{-1})^*\widetilde{\alpha} &= \eta^*(\widehat{f}_x^{-1})^*(\widehat{f}_x^*g)|_{\widetilde{\Gamma}_{f^{-1}x}\setminus\Gamma_{f^{-1}x}}\circ\widehat{f}_x^*(\eta^{-1})^*\widetilde{\alpha}\\
&= \eta^*g|_{\widetilde{\Gamma}_{f^{-1}x}\setminus\Gamma_{f^{-1}x}}\circ(\eta^{-1})^*\widetilde{\alpha}.
\end{aligned}$$

The analogous statement holds for the (N,m)-truncated objects by an identical argument. $\square$

Theorem 4.19. *Let D be a metric disk in $\mathbb{C}$. There exists a stratified map*

$$\Psi^{\mathrm{equiv,Ran}}_{[0,1]}:[0,1]\times\left(\mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(\mathbb{A}^1_{\mathbb{C}})}\times_{\mathrm{Ran}(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}}\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(\mathbb{A}^1_{\mathbb{C}})}\right)\to \mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(\mathbb{A}^1_{\mathbb{C}})}\times_{\mathrm{Ran}(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}}\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(\mathbb{A}^1_{\mathbb{C}})}$$

such that

1. *for any $t\in[0,1]$, $\Psi^{\mathrm{equiv,Ran}}_t$ is a closed embedding;*
2. *the square*

$$\begin{tikzcd}
{[0,1]}\times\mathrm{L}^+G_{\mathrm{Ran}(D)}\times_{\mathrm{Ran}(D)}\mathrm{Gr}_{G,\mathrm{Ran}(D)} \arrow[r, "\Psi^{\mathrm{equiv,Ran}}_{[0,1]}|_{\mathrm{Gr}_{G,\mathrm{Ran}(D)}}"] \arrow[d, "\mathrm{id}\times\mathrm{act}_{\mathrm{Ran}(D)}"] & \mathrm{L}^+G_{\mathrm{Ran}(D)}\times_{\mathrm{Ran}(D)}\mathrm{Gr}_{G,\mathrm{Ran}(D)} \arrow[d, "\mathrm{act}_{\mathrm{Ran}(D)}"] \\
{[0,1]}\times\mathrm{Gr}_{G,\mathrm{Ran}(D)} \arrow[r, "\Psi^{\mathrm{Ran}}_{[0,1]}|_{\mathrm{Gr}_{G,\mathrm{Ran}(D)}}"] & \mathrm{Gr}_{G,\mathrm{Ran}(D)},
\end{tikzcd}$$

   where $\mathrm{act}_{\mathrm{Ran}(D)}$ is the action map, commutes.

Proof. The only difference with respect to the previous proof is that one builds the map $\Psi^{\mathrm{equiv,Ran}}_{[0,1]}$ in the same way as Theorem 4.1, by filtering $\mathrm{Gr}^{\mathrm{an}}_{G,\mathrm{Ran}(X)}$ and then inducing maps on perfect quotients. Therefore, by construction, $\Psi^{\mathrm{equiv,Ran}}_{[0,1]}$ agrees with the action of $\mathrm{L}^+G^{\mathrm{an}}_{\mathrm{Ran}(X)}$. $\square$

Remark 4.20. A nice way to rephrase the Theorem 4.19 is the following. One can form a stratified topological stack defined as the quotient stack, relative to $\mathrm{Ran}(D)$,

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Ran}(D)} = \mathrm{Gr}_{G,\mathrm{Ran}(D)}/\mathrm{L}^+G_{\mathrm{Ran}(D)}$$

for any metric disk, and then use Theorem 4.19 to prove that the induced embedding

$$\mathcal{H}\mathrm{ck}_{G,\mathrm{Ran}(D')} \to \mathcal{H}\mathrm{ck}_{G,\mathrm{Ran}(D)}$$

is a stratified homotopy equivalence of stacks. We chose not to delve into this formalism in the present paper, but the reader can find all the needed terminology in [Noc20, Appendix B.3], [Jan23].

4.4 $\mathbb{E}_2$-algebra structure

The aim of this final subsection is to prove Corollary B.

Recall 4.21. Let Fin_* be the category of pointed finite sets, and $N : \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_\infty$ the simplicial nerve.

We say that a functor of ∞-categories $p : \mathcal{O}^\otimes \to N(\mathrm{Fin}_*)$ is an ∞-*operad* if it satisfies the conditions of [Lur17, Definition 2.1.1.10].

A *map of* ∞-*operads* $A : \mathcal{O}^\otimes \to \mathcal{O}'^\otimes$ is a functor over $N(\mathrm{Fin}_*)$ satisfying the conditions of [Lur17, Definition 2.1.2.7].

Recall 4.22. [Lur17, Definition 5.4.4.1] Let Surj denote the subcategory of Fin_* spanned by all objects of Fin_*, but only surjective maps. If $\mathcal{O}^\otimes$ is an ∞-operad, let $\mathcal{O}^\otimes_{\mathrm{nu}}$ be its sub-∞-category with the same objects and as morphisms those whose image via $p : \mathcal{O}^\otimes \to N(\mathrm{Fin}_*)$ lie in $N(\mathrm{Surj})$. The map to $N(\mathrm{Fin}_*)$ inherited by inclusion in $\mathcal{O}^\otimes$ exhibits $\mathcal{O}^\otimes_{\mathrm{nu}}$ as a sub-∞-operad of $\mathcal{O}^\otimes$.

The ∞-operad $\mathcal{O}^\otimes_{\mathrm{nu}}$ is called the *non-unital* version of $\mathcal{O}^\otimes$.

Recall 4.23. Let $\mathcal{C}$ be a category with finite products. There is a Cartesian symmetric monoidal structure (in the sense of [Lur17, §2.4.1]) $q : N(\mathcal{C})^\times \to N(\mathrm{Fin}_*)$ on the simplicial nerve $N(\mathcal{C})$: $N(\mathcal{C})$ is recovered as is the fiber $q^{-1}(\langle 1\rangle)$, and more generally the fiber over $\langle n\rangle$ is equivalent to $N(\mathcal{C})^{\times n}$.

Recall 4.24. Let $p : \mathcal{O}^\otimes \to \mathrm{Fin}_*$ be an operad. An $\mathcal{O}^\otimes$-*algebra object (with values) in* $\mathcal{C}^\times$ is a map of operads $A : \mathcal{O}^\otimes \to N(\mathcal{C}^\times)$. These form an ∞-category $\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C}^\times)$. A *non-unital* $\mathcal{O}^\otimes$-algebra object in $\mathcal{C}^\times$ is a $\mathcal{O}^\otimes_{\mathrm{nu}}$-algebra object in $\mathcal{C}^\times$.

A non-unital $\mathcal{O}^\otimes$-algebra object A in $\mathcal{C}^\times$ is *locally constant* if every morphism in $\mathcal{O}^\otimes_{\mathrm{nu}}$ lying in the fiber over $\langle 1\rangle \in N(\mathrm{Fin}_*)$ is sent to an isomorphism under A.

Remark 4.25. The terminology may be misleading: the category $\mathrm{Alg}_{\mathcal{O}^\otimes_{\mathrm{nu}}}(\mathcal{C}^\times)$, of non-unital $\mathcal{O}^\otimes$-algebra objects is not a subcategory of $\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C}^\times)$: on the contrary, there is a restriction functor

$$\mathrm{Alg}_{\mathcal{O}^\otimes}(\mathcal{C}^\times) \to \mathrm{Alg}_{\mathcal{O}^\otimes_{\mathrm{nu}}}(\mathcal{C}^\times).$$

Notation 4.26. Let $\mathrm{Disk}(\mathbb{R}^2)$ be the category of open subsets $U \subset \mathbb{R}^2$ homeomorphic to $\mathbb{R}^2$, where morphisms are the inclusions. Let $\mathrm{MDisk}(\mathbb{R}^2)$ be the full subcategory of metric disks $D \subset \mathbb{R}^2$. Let $\mathrm{Disk}(\mathbb{R}^2)^\otimes$ be the category whose objects are tuples of opens $(U_1, \ldots, U_n)$ and whose morphisms $(U'_1, \ldots, U'_m) \to (U_1, \ldots, U_n)$ consist to order-preserving maps $d : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that

$$\forall\ 1 \le i < j \le m \text{ s.t. } d(i) = d(j) = k \implies U'_i, U'_j \subset U_k \ \ \&\ \ U'_i \cap U'_j \neq \varnothing.$$

Denote by $\mathrm{MDisk}(\mathbb{R}^2)^\otimes$ the full subcategory of $\mathrm{Disk}(\mathbb{R}^2)^\otimes$ spanned by tuples of metric disks $(D_1, \ldots, D_n)$.

Lemma 4.27. *$N(\mathrm{Disk}(\mathbb{R}^2)^{\otimes})$ and $N(\mathrm{MDisk}(\mathbb{R}^2)^{\otimes})$ have a structure of ∞-operads.*

Proof. There is a natural map $\mathrm{Disk}(\mathbb{R}^2)^{\otimes} \to \mathrm{Fin}_*$ which sends $(U_1, \dots, U_n) \mapsto \langle n \rangle = \{*, 1, \dots, n\}$. Taking the simplicial nerve of it we get

$$N(\mathrm{Disk}(\mathbb{R}^2)^{\otimes}) \to N(\mathrm{Fin}_*).$$

Notice that $N(\mathrm{Disk}(\mathbb{R}^2)^{\otimes})$ is the same as the ∞-category $N(\mathrm{Disk}(\mathbb{R}^2))^{\otimes}$ described by Lurie in [Lur17, Remark 5.4.5.6]. We conclude by using [Lur17, Remark 5.4.5.7]. □

Remark 4.28. Since $\mathrm{Disk}(\mathbb{R}^2)$ is a category over Fin_*, one can define $\mathrm{Disk}(\mathbb{R}^2)_{\mathrm{nu}}$ to be the subcategory defined as the fiber product $\mathrm{Disk}(\mathbb{R}^2) \times_{\mathrm{Fin}_*} \mathrm{Surj}$. Notice that the nerve of $\mathrm{Disk}(\mathbb{R}^2)_{\mathrm{nu}}$ coincides with $N(\mathrm{Disk}(\mathbb{R}^2)^{\otimes})_{\mathrm{nu}}$. Same definition and property holds for $\mathrm{MDisk}(\mathbb{R}^2)$.

Recall 4.29. Recall the definition of the *little 2-disks ∞-operad* $\mathbb{E}_2$ from [Lur17, Definition 5.1.0.2]. Unlike $\mathrm{Disk}(\mathbb{R}^2)^{\otimes}$ and $\mathrm{MDisk}(\mathbb{R}^2)^{\otimes}$, $\mathbb{E}_2$ is not the nerve of a 1-category[11].

Recall 4.30. By [Lur17, Theorem 5.4.5.15] there is an equivalence between the category of non-unital $\mathbb{E}_2$-algebra objects in $\mathcal{C}^{\times}$ and the category of locally constant non-unital $\mathrm{Disk}(\mathbb{R}^2)^{\otimes}$-algebra objects in $\mathcal{C}^{\times}$.

The principle in Recall 4.30 is the main tool of the present subsection. However, we will need a slightly modified version.

Proposition 4.31. *There is an equivalence between the category of non-unital $\mathbb{E}_2$-algebra objects in $\mathcal{C}^{\times}$ and the category of locally constant non-unital $\mathrm{MDisk}(\mathbb{R}^2)^{\otimes}$-algebra objects in $\mathcal{C}^{\times}$.*

Proof. The aforementioned [Lur17, Theorem 5.4.5.15] rests upon [Lur17, Lemma 5.4.5.10, Lemma 5.4.5.11]. Both lemmas hold if one replaces $\mathrm{Disk}(\mathbb{R}^2)^{\otimes}$ with $\mathrm{MDisk}(\mathbb{R}^2)^{\otimes}$: indeed, they rely on the categorical Seifert-Van Kampen Theorem [Lur17, Theorem A.3.1], and therefore one can consider a base of the topology of $\mathbb{C}$ instead of all disks. This means that [Lur17, Theorem 5.4.5.15] holds with $\mathrm{MDisk}(\mathbb{R}^2)^{\otimes}_{\mathrm{nu}}$ in place of $\mathrm{Disk}(\mathbb{R}^2)^{\otimes}_{\mathrm{nu}}$, and we can conclude. □

Theorem 4.32. *The functor*

$$\mathrm{Gr}_{G,\mathrm{Ran}(-)} : \mathrm{MDisk}(\mathbb{R}^2) \to \mathrm{StrTop}[W^{-1}], \qquad D \mapsto \mathrm{Gr}_{G,\mathrm{Ran}(D)}$$

upgrades to a locally constant non-unital $\mathrm{MDisk}(\mathbb{R}^2)^{\otimes}$-algebra object A in $\mathrm{StrTop}[W^{-1}]^{\times}$. As a consequence, when any $D \in \mathrm{MDisk}(\mathbb{R}^2)$ is fixed, $\mathrm{Gr}_{G,\mathrm{Ran}(D)}$ carries a non-unital $\mathbb{E}_2$-algebra structure in $\mathrm{StrTop}[W^{-1}]^{\times}$, and this structure is independent of the choice of D.

Proof. On objects, we define our functor as

$$A : \mathrm{MDisk}(\mathbb{R}^2)^{\otimes}_{\mathrm{nu}} \to \mathrm{StrTop}[W^{-1}]^{\times}, \quad (D_1, \dots, D_k) \mapsto \prod_{i=1}^{k} \mathrm{Gr}_{G,\mathrm{Ran}(D_i)}.$$

On morphisms, we define it in two special cases, from which the general definition and functoriality are easily deduced:

---

[11] Its objects are the same as Fin_*, but $\mathrm{Hom}_{\mathbb{E}_2}(\langle m \rangle, \langle n \rangle)$ is the *space* (not just a set)

$$\coprod_{f:\langle m \rangle \to \langle n \rangle \text{ in } \mathrm{Fin}_*} \prod_{j=1}^{n} \mathrm{Rect}((-1,1)^2 \times f^{-1}(\{j\}), (-1,1)^2)$$

where $(-1,1)$ is the interval in $\mathbb{R}$ and Rect stays for the space of *rectilinear embeddings* (see *loc. cit.*).

- binary *inert* morphisms. Let D_1, D_2 be two disks (disjoint or not) and $\alpha : (D_1, D_2) \to (D_1)$ the inert map lying over

$$\langle 2 \rangle \to \langle 1 \rangle, \quad *, 2 \mapsto * \text{ and } 1 \mapsto 1$$

  We define $A(\alpha)$ as the projection $\mathrm{Gr}_{G,\mathrm{Ran}(D_1)} \times \mathrm{Gr}_{G,\mathrm{Ran}(D_2)} \to \mathrm{Gr}_{G,\mathrm{Ran}(D_1)}$.

- binary *active* morphisms. Let D'_1, D'_2 be two disjoint metric disks contained in another metric disk D, corresponding to a map $\alpha : (D_1, D_2) \to D$ in $\mathrm{MDisk}(\mathbb{R}^2)^{\otimes}_{\mathrm{nu}}$, lying over the map

$$\langle 2 \rangle \to \langle 1 \rangle, \quad * \mapsto * \text{ and } 1, 2 \mapsto 1$$

  in Fin_*. Let $n \geq 1, I, J \in \mathrm{Fin}_{\geq 1,\mathrm{surj}}$ s.t. $|I|, |J| \leq n$. Let $(\mathbb{C}^I \times \mathbb{C}^J)_{\mathrm{disj}} \subset \mathbb{C}^I \times \mathbb{C}^J$ be the open of pairs of sequences $(x, x') = ((x_1, \ldots, x_{|I|}), (x'_1, \ldots, x'_{|J|}))$ such that $\Gamma_x \cap \Gamma_{x'} = \varnothing$. By using the factorization property (3.16) and then analytifying, we can consider the map

$$(\mathrm{Gr}^{(N)}_{G,\mathbb{C}^I} \times \mathrm{Gr}^{(N)}_{G,\mathbb{C}^J})_{\mathrm{disj}} = (\mathrm{Gr}^{(N)}_{G,\mathbb{C}^I} \times \mathrm{Gr}^{(N)}_{G,\mathbb{C}^J}) \times_{(\mathbb{C}^I \times \mathbb{C}^J)} (\mathbb{C}^I \times \mathbb{C}^J)_{\mathrm{disj}} \to \mathrm{Gr}^{(N)}_{G,\mathbb{C}^{I \amalg J}}$$

  This coincides with taking the union of systems of points and gluing the torsors along the complement of that union. Since this map is the analytification of the analogously defined functor between $\mathrm{Gr}_{G,X^I\phi} \times \mathrm{Gr}_{G,X^\psi}$, it is continuous and therefore we obtain

$$\mathrm{Gr}_{G,(D'_1)^I} \times \mathrm{Gr}_{G,(D'_2)^J} \to \mathrm{Gr}_{G,D^{I \amalg J}} \to \mathrm{Gr}_{G,\mathrm{Ran}_{\leq 2n}(D)}. \tag{4.13}$$

  Recall from the proof of Theorem 4.1 that the canonical map

$$\mathrm{Gr}_{G,(D'_i)^I} \to \mathrm{Gr}_{G,\mathrm{Ran}_{\leq n}(D'_i)}$$

  is a closed quotient by the relation

$$(x_1, \ldots, x_{|I|}, \mathcal{F}, \alpha) \sim (x'_1, \ldots, x'_{|I|}, \mathcal{F}', \alpha') \iff \{x_1, \ldots, x_{|I|}\} = \{x'_1, \ldots, x'_{|I|}\}, \mathcal{F} \simeq \mathcal{F}', \alpha \simeq \alpha'.$$

  Therefore, the map (4.13) factors through this closed quotient and yields a continuous map

$$\mathrm{Gr}_{G,\mathrm{Ran}_{\leq n}(D'_1)} \times \mathrm{Gr}_{G,\mathrm{Ran}_{\leq n}(D'_2)} \to \mathrm{Gr}_{G,\mathrm{Ran}_{\leq 2n}(D)}.$$

  We can now use [Har15] just like in (4.4) and obtain a continuous map

$$\mathrm{Gr}^{\otimes}_{G,\mathrm{Ran}(-)}(\alpha) : \mathrm{Gr}_{G,\mathrm{Ran}(D'_1)} \times \mathrm{Gr}_{G,\mathrm{Ran}(D'_2)} \to \mathrm{Gr}_{G,\mathrm{Ran}(D)}.$$

Note that, by the first point of the above definition, A is a map of operads. One can easily see that the functor $\mathrm{Gr}^{\otimes}_{G,\mathrm{Ran}(-)}$ is a map of operads. We now observe that it is locally constant in the sense of Recall 4.24, because for $D' \subset D$ metric disks, Corollary 4.15 tells us that the induced map $\mathrm{Gr}_{\mathrm{Ran}(D')} \hookrightarrow \mathrm{Gr}_{\mathrm{Ran}(D)}$ is a stratified homotopy equivalence. The application of Proposition 4.31 concludes the proof.

The underlying stratified space (up to stratified homotopy equivalence) of our algebra object is given by the value $\mathrm{Gr}_{\mathrm{Ran}(D_0)}$, where the choice of $D_0 \in \mathrm{MDisk}(\mathbb{R}^2)$ is irrelevant (the values are all stratified homotopy equivalent). □

Remark 4.33. In the setting of stratified topological stacks mentioned in Remark 4.20, one can prove in the same way an analogous statement involving the $\mathrm{Hck}_{\mathrm{Ran}(D)}$'s, by means of Corollary 4.15 and Remark 4.20.

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