(*) Resume & finish Laplace transform discussion (bits from §6.3, 6.4, 6.5, 6.6)

(*) Review for the Final.

Idea: given a scalar IVP : LHS = RHS : apply $\mathcal{L}(-)$ on both sides $\mathcal{L}(LHS) = \mathcal{L}(RHS)$

Solve for $\mathcal{L}(y)$ & then find the Laplace inverse for $\mathcal{L}(y)$.

Example: $\begin{cases} y''' + y'' + y' + y = 0 \\ y(0) = 1, \ y'(0) = 0, \ y''(0) = 1. \end{cases}$ ← ODE : III order : then assume y-solution-satisfy the hp's of the thm about $\mathcal{L}(y^{(n)})$ with $n = 1, 2, 3$

↑ at the end we need to check this.

$\Rightarrow$ apply $\mathcal{L}(-)$: $\mathcal{L}(y''') + \mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = 0$.

HP: y, y', y'' continuous + y''' piecewise cont. + $\exists$ (a, K, M) real constants s.t. $|y|$, $|y'|$ & $|y''| \leq Ke^{at}$ $\forall t \geq M$

THEN $\mathcal{L}(y')$, $\mathcal{L}(y'')$, $\mathcal{L}(y''')$ $\exists$ $\forall s > a$ & we have the nice formulas:

$\mathcal{L}(y''') = s^3\mathcal{L}(y) - s^2 y(0) - sy'(0) - y''(0) = s^3\mathcal{L}(y) - s^2 - 1$;

$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - s$; $\mathcal{L}(y') = s\mathcal{L}(y) - y(0) = s\mathcal{L}(y) - 1$ $\Bigg]$ $\forall s > a$

$\Rightarrow LHS = (s^3 + s^2 + s + 1)\mathcal{L}(y) + (-s^2 - 1 - s - 1) = 0 \Rightarrow \mathcal{L}(y) = \dfrac{s^2+s+2}{s^2(s+1) + (s+1)} = \dfrac{s^2+s+2}{(s^2+1)(s+1)} = \dfrac{(s^2+1)+(s+1)}{(s^2+1)(s+1)} =$

$= \dfrac{1}{s+1} + \dfrac{1}{s^2+1}$.

Rule: trick: $\dfrac{s^2+s+2}{(s^2+1)(s+1)}$ = try to rewrite it as: $\dfrac{As+B}{(s^2+1)} + \dfrac{C}{s+1}$ (we want $p_1(t) \cdot As+B$ to have the same degree as $C\,p_2(t)$).

polynomial of degree 2

polynomial of degree 2 $p_2(t)$

polynomial of degree 1 $p_1(t)$

degree 1 & not zero.

$p_2(t)$ $p_1(t)$

In this cases $(As+B)(s+1) + C(s^2+1) = s^2+s+2$

"

$As^2 + Bs + As + B + Cs^2 + C = s^2 + s + 2$: $A + C = 1$ $\Rightarrow$ $2A + B + C = 2$ $\Rightarrow$ $\underline{A = 0}$

$B + A = 1$ $\Rightarrow$ $\underline{C = B = 1}$

$B + C = 2$

$\Rightarrow \dfrac{1}{s^2+1} + \dfrac{1}{s+1}$ ✓.

$\mathcal{L}(y) = \dfrac{1}{s^2+1} + \dfrac{1}{s+1} = \underset{\forall s>0}{\mathcal{L}(\sin(t))} + \underset{\forall s>-1}{\mathcal{L}(e^{-t})} = \mathcal{L}(\sin(t) + e^{-t}) \Rightarrow \boxed{y = \sin(t) + e^{-t}}$.

use the table

it is C^∞ &

(*) $|\sin(t) + e^{-t}| \leq 2 \quad \forall t \geq 0$.

(*) similar for the derivatives.

Rule: The same procedure works for linear ODE with constant coeff.

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(t)$$

(*) Apply $\mathcal{L}$ on both sides & ensure the solution satisfies hp's of regularity in order to have the formula for $\mathcal{L}(y^{(n)})$.

Ex: $\begin{cases} y'' + y = \sin(2t) \\ y(0) = 2,\ y'(0) = 1 \end{cases}$. Assume y, y' continuous & y'' piecewise & $|y|, |y'| \le ke^{at}$ $\forall t \ge M$ for some (a, k, M).

LHS : $s > a$.

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) = s^2\mathcal{L}(y) - 2s - 1$$

RHS

$$\mathcal{L}(\sin(2t)) = \frac{2}{s^2+4} \quad s > 0.$$

$$\Rightarrow s^2\mathcal{L}(y) + \mathcal{L}(y) = 2s + 1 + \frac{2}{s^2+4} \Rightarrow \mathcal{L}(y) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+4)(s^2+1)}$$

$$\Rightarrow \frac{2s}{s^2+1} + \frac{1}{s^2+1} + \frac{2}{3}\frac{s^2+4-(s^2+1)}{(s^2+4)(s^2+1)} = 2\,\mathcal{L}(\cos(t)) + \mathcal{L}(\sin(t)) + \frac{2}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right]$$

$$= \mathcal{L}(2\cos(t) + \sin(t)) + \frac{2}{3}\left(\mathcal{L}(\sin(t)) - \frac{1}{2}\sin(2t)\right)$$

$$= \mathcal{L}\left(2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)\right)$$

$$\Rightarrow y(t) = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t).$$ $\leftarrow C^\infty$ & bounded: initial hp's satisfied ✓.

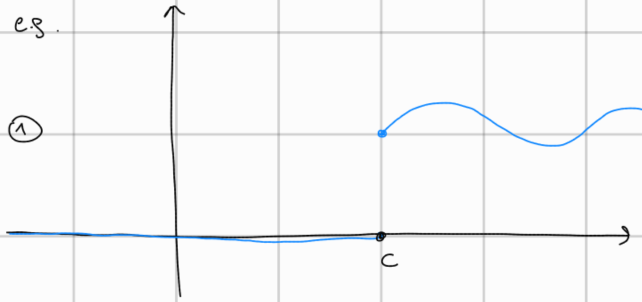
Rmk: however so far we have treated IVP that we could have solved by other means (undetermined coeff.)

Let's see now examples where $g(t)$ is discontinuous or an impulsive function. This occurs in physics when a system is perturbed by a forcing function after a period of time.

This means that the function $g(t)$ is piecewise continuous.

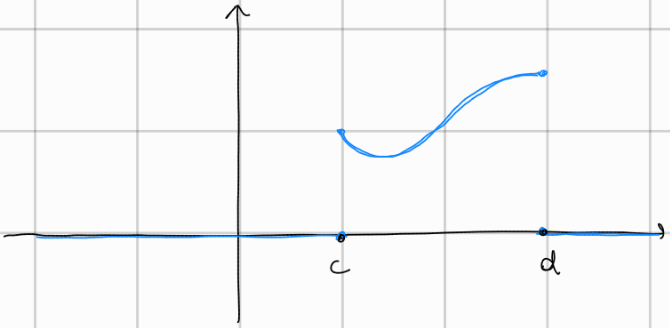
e.g.

1



← $g(t)$ was zero up to time c

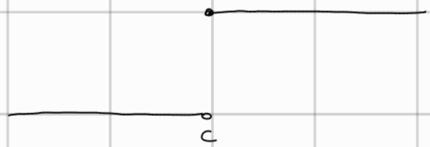
2



← $g(t)$ is not zero between c, d

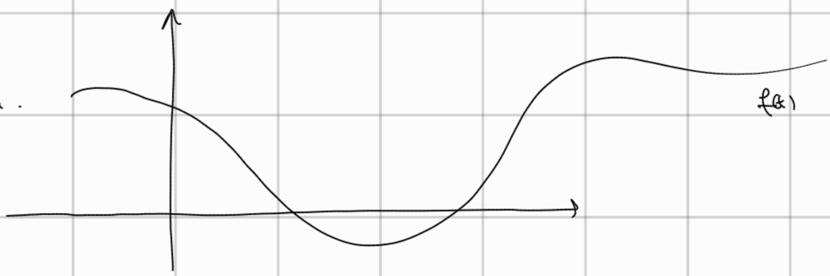
To deal with piecewise continuous function is very useful to introduce the unit step function.

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases} \Rightarrow$$

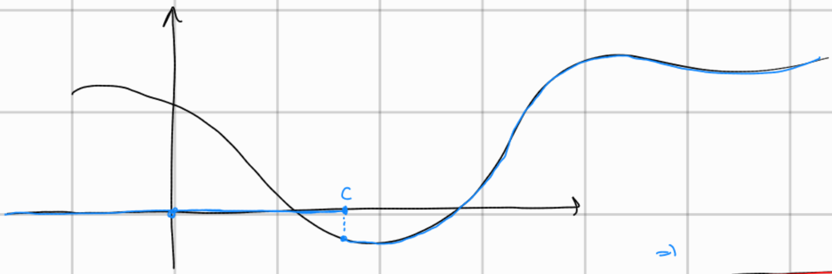


⇐ we can use it to build piecewise continuous functions.

Indeed if $f(t)$ is a continuous function.



$\Rightarrow f(t)u_c(t) \Rightarrow$

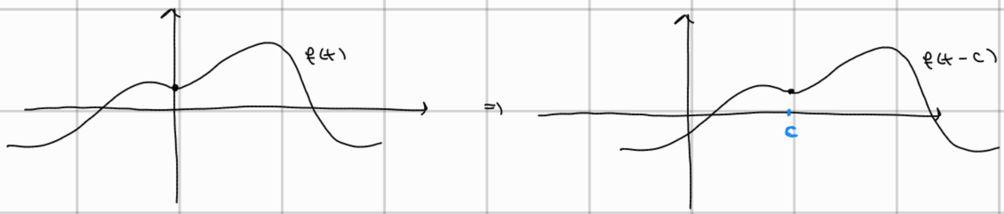


$\Rightarrow$

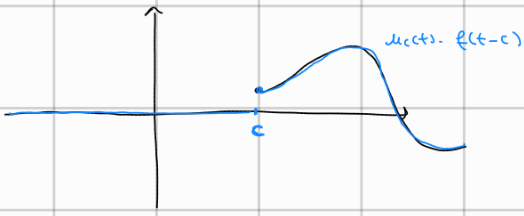
Thm: Assume $\mathcal{L}(f(t))$ $\exists$ for $s > a \geq 0$ & $c > 0$. Then

$$\mathcal{L}(u_c(t) \cdot f(t-c)) = e^{-cs}\mathcal{L}(f(t)) \quad \exists \text{ for } s > a.$$

Rmk: we need to take the translate of $f(t)$: $t \to t-c$ $\Rightarrow$ the graph of f is pushed forward of c :



$\Rightarrow u_c(t) \cdot f(t-c)$ $\Rightarrow$



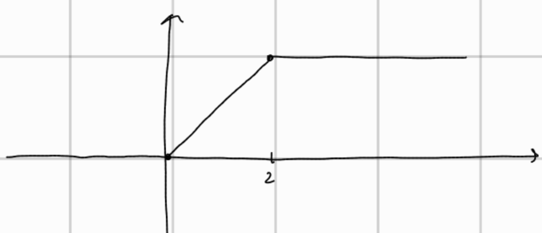
Ex: $g(t) = \begin{cases} 0 & t < \pi/4 \\ \cos(t - \pi/4) & t \geq \frac{\pi}{4} \end{cases}$ $\Rightarrow g(t) = u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4})$ $\Rightarrow \mathcal{L}(g(t)) = e^{-\frac{\pi}{4}s}\mathcal{L}(\cos(t))$

$$= e^{-\frac{\pi}{4}s}\frac{s}{s^2+1}$$

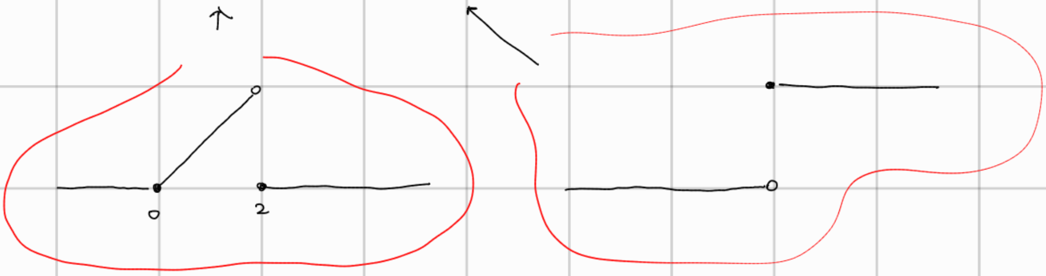
Rmk: what if we want to consider a function $g(t)$ only on $[c, d]$ $d \in \mathbb{R}$ and not on $[c, +\infty)$.

$$\Rightarrow g(t)[u_c(t) - u_d(t)] : \begin{cases} 0 & t < c \\ g(t) & c \leq t < d \\ 0 & t \geq d \end{cases}$$

Ex: $g(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 2 \\ 2 & t \geq 2 \end{cases}$



$\Rightarrow \quad t\cdot(u_0(t) - u_2(t)) + 2u_2(t)$



$$\Rightarrow \mathcal{L}(u_0(t)\cdot t - u_2(t)\cdot t + 2u_2(t)) = \mathcal{L}(u_0(t)\cdot t - u_2(t)(t-2))$$

$$= \mathcal{L}(t) - \mathcal{L}(u_2(t)(t-2))$$

$$= \frac{1}{s^2} - e^{-2s}\frac{1}{s^2}$$

This shift $t \to t-c$ is there also in another case.

Thm: if $\mathcal{L}(f)$ $\exists$ for $s>a$ then

$$\mathcal{L}(e^{ct}\cdot f(t)) = \mathcal{L}(f)(s-c) \quad \text{for } s>a+c$$

it is $\mathcal{L}(f)(s)$ evaluated in $s-c$, not a product.

Ex: $G(s) = \dfrac{1}{s^2-4s+5} = \dfrac{1}{(s-2)^2+1}$. $\qquad \mathcal{L}(\sin(t))(s) = \dfrac{1}{s^2+1}$

$$\| $$

$$\mathcal{L}(\sin(t))(s-2) = \mathcal{L}(e^{2t}\cdot\sin(t)) \Rightarrow \mathcal{L}^{-1}(G(s)) = e^{2t}\sin(t)$$

Example: $\begin{cases} y'' - 3y' + 2y = g(t) \\ y(0) = y'(0) = 0 \end{cases}$ & $g(t) = \begin{cases} 0 & 0 \le t < 5 \\ 1 & 5 \le t < 20 \\ 0 & t \ge 0 \end{cases}$ $\Rightarrow g(t) = u_5(t) - u_{20}(t)$

Assuming: y, y' continuous & y'' piecewise continuous & $|y|, |y'| \le Ke^{at}$ $\forall t \ge M$.

$\Rightarrow \mathcal{L}(y'') = s^2\mathcal{L}(y)$, $\mathcal{L}(y') = s\mathcal{L}(y)$ $\Rightarrow (s^2 - 3s + 2)\mathcal{L}(y) = \mathcal{L}(u_5(t)) - \mathcal{L}(u_{20}(t))$

$$= \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$\Rightarrow \mathcal{L}(y) = \dfrac{(e^{-5s} - e^{-20s})}{s(s-2)(s-1)} = \dfrac{e^{-5s}}{s(s-2)(s-1)} - \dfrac{e^{-20s}}{s(s-2)(s-1)}$ Call $F(s) = \dfrac{1}{s(s-2)(s-1)} = \mathcal{L}(f(t))(s)$

Use $\mathcal{L}(u_c(t) \cdot f(t-c)) = e^{-cs}\mathcal{L}(f(t))$ $\Rightarrow = \mathcal{L}(u_5(t) \cdot f(t-5) - u_{20}(t) f(t-20))$

$$\frac{1}{s(s-2)(s-1)} = \frac{(s-1)-(s-2)}{s(s-2)(s-1)} = \frac{1}{s(s-2)} - \frac{1}{s(s-1)} = \frac{1}{2}\frac{s-(s-2)}{s(s-2)} - \frac{s-(s-1)}{s(s-1)} = \frac{1}{2}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s} - \frac{1}{s-1} + \frac{1}{s}$$

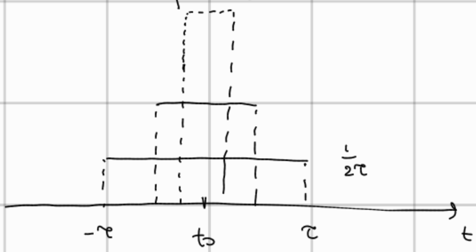
$$= \frac{1}{2}\frac{1}{s-2} + \frac{1}{2}\frac{1}{s} - \frac{1}{s-1} = F(s)$$

$\Rightarrow f(t) = \frac{1}{2}e^{2t} + \frac{1}{2} - e^t$ $\Rightarrow y(t) = u_5(t)\left[\frac{1}{2}e^{2(t-5)} + \frac{1}{2} - e^{t-5}\right] - u_{20}(t)\left[\frac{1}{2}e^{2(t-20)} + \frac{1}{2} - e^{t-20}\right]$

Wilder situations

"you have a particle which moves accordingly to $ay''+by'+cy=>$ until time t_0 at which point the particle receives an impulse in an infinitesimally short amount of time"

$\Rightarrow$ the model of this looks like the limit of



$$\Rightarrow \delta(t) = \lim_{\tau\to 0} d_\tau(t) \quad \text{where}$$

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau} & -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases}$$

Remk.: $\displaystyle\int_{-\infty}^{+\infty} d_\tau(t) = 1 \quad \forall \tau$

So the equation becomes: $ay''+by'+cy = \delta(t-t_0)$.

Let's compute $\mathcal{L}(\delta(t-t_0))$. First compute $\mathcal{L}(d_\tau(t)) = \displaystyle\int_0^{+\infty} e^{-st}\cdot\frac{1}{2\tau}(u_{(t_0-\tau)}(t) - u_{(t_0+\tau)}(t)) =$

$$= \int_{t_0-\tau}^{t_0+\tau} e^{-st}\cdot\frac{1}{2\tau}\,dt = -\frac{1}{2\tau s}e^{-st}\Bigg|_{t_0-\tau}^{t_0+\tau} = -\frac{1}{2\tau s}[e^{-s(t_0+\tau)} - e^{-s(t_0-\tau)}] = -\frac{e^{-t_0 s}}{2\tau s}\cdot(e^{-s\tau}-e^{s\tau})$$

Take the limit: $\displaystyle\lim_{\tau\to 0}\frac{e^{-t_0 s}}{2s}\left[\frac{e^{s\tau}-e^{-s\tau}}{\tau}\right] \underset{\uparrow}{=} \frac{e^{-t_0 s}}{2s}\lim_{\tau\to 0}\frac{se^{s\tau}+se^{s\tau}}{1} = e^{-t_0 s}$

de L' Hospital

$$\Rightarrow \mathcal{L}(\delta(t-t_0)) = e^{-t_0 s}.$$

Example: $2y''+y'+2y = \delta(t-3) \Rightarrow 2\mathcal{L}(y'') + \mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(t-3) = e^{-3s}$

Convolution thm: If $F(s) = \mathcal{L}(f(t))$ & $G(s) = \mathcal{L}(g(t))$ both $\exists$ for $s > a \geq 0$, then

$$H(s) = F(s)\,G(s) = \mathcal{L}(h(t)) \qquad s > a \qquad \text{where}$$

$$h(t) = \int_0^t f(t-s)\,g(s)\,ds = \int_0^t f(s)\,g(t-s)\,ds .$$

Example: $\frac{a}{s^2(s^2+a^2)} = H(s)$. Find $h(t)$: $\mathcal{L}(h(t)) = H(s)$.

(1sT) $$\frac{(s^2+a^2)-s^2}{a\,s^2\,(s^2+a^2)} = \frac{1}{as^2} - \frac{1}{a(s^2+a^2)} = \mathcal{L}\left(\frac{t}{a}\right) - \frac{1}{a^2}\left(\frac{a}{s^2+a^2}\right) = \mathcal{L}\left(\frac{t}{a}\right) - \frac{1}{a^2}\mathcal{L}\left(\sin(at)\right)$$

$$\Rightarrow h(t) = \frac{t}{a} - \frac{\sin(at)}{a^2}$$

(2nd) $F(s) = \frac{1}{s^2}$, $G(s) = \frac{a}{s^2+a^2}$

$\qquad\quad \| \qquad\qquad\qquad \|$

$\quad \mathcal{L}(t) \qquad\qquad \mathcal{L}(\sin(at))$

$$\Rightarrow \int_0^t (t-s)\sin(as)\,ds = \frac{t}{a}\left[-\cos(as)\right]_0^t - \int_0^t s\cdot\sin(as)\,ds$$

$$= \frac{t}{a}\left[-\cos(at)+1\right] - \left[-\frac{s\cos(as)}{a}\Big|_0^t + \int_0^t \frac{\cos(as)}{a}\,ds \right]$$

$$= -\frac{t}{a}\cancel{\cos(at)} + \frac{t}{a} + \frac{t\cos(at)}{\cancel{a}} - \left[\frac{\sin(as)}{a^2}\right]_0^t$$

$$= \frac{t}{a} - \frac{\sin(at)}{a^2}.$$

REVIEW for the Final: After the midterm we have discussed:

* general solution for the hom. linear system $x' = Ax$, A constant
* general solution for the scalar higher order ODE, constant coeff.

* given a fundamental set of solution for the homog. case:
  * particular solution for $x' = A(t)x + b(t)$
  * particular solution for $y^{(n)} + \dots + a_0 y = g(t)$

* Power series method for $P(t)y'' + Q(t)y' + R(t)y = 0$ around t_0 ordinary
  for $t > t_0$ reg. singular

* Qualitative discussion for $x' = Ax$ & Linearization of $x' = G(x)$ autonomous non-linear systems.

* Laplace transform as alternative method / for non-continuous known functions $g(t)$.

Final: 2 problems on the first half (pre-midterm)

6 problems on the second half:
* one on Laplace transform
* one non-linear systems (qualitative discussion)
* one power series problem

⊛ one (homogeneous) linear system (qualitative discussion with pictures).

\+ particular solution for $x' = Ax + b(t)$.

⊛ one higher-order scalar linear with constant coeff's

⊛ one $x' = Ax$ system - find general solution.

REVIEW for analytic solution of $x' = AX$.

⊛ First step: <u>eigenvalues</u>: $\det(A - \lambda \mathrm{Id}) = 0$ with their algebraic multiplicity.

⊛ Second step: <u>eigenvector</u>: solve $(A - \lambda \mathrm{Id})v = 0$.

⊛ Third step (if necessary) ⊙ find other solutions if $\mu_a > \mu_g$ e.g. $(w + tv)e^{\lambda t}$.

⊙ put the solution in real form (in the case of $\lambda, \bar{\lambda}$ - complex).

<u>Example</u>: $x' = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix} x$: $\det(A - \lambda \mathrm{Id}) = \det \begin{bmatrix} -1-\lambda & 1 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix} \cdot \det \begin{bmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{bmatrix} =$

$$= -(\lambda+1)^3 [(1-\lambda)^2 + 4] = -(\lambda+1)^3 [\lambda^2 - 2\lambda + 5] \Rightarrow \begin{cases} \lambda = -1 & \mu_a = 3 \\ \lambda_{1,2} = 1 \pm \sqrt{1-5} = 1 \pm 2i \end{cases}$$

$\lambda = -1$: $A + \mathrm{Id} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 2 & 2 \end{pmatrix} \Rightarrow v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ enough $v_2 = 0$: $\begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ span $\begin{matrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}$ & $\begin{matrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{matrix}$

I can get Id because $\lambda = -1$ is NOT an eigenvalue for $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$\mu_g = 2$

Need extra solution:
$$\left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \hline & 0 & & 2 & -2 \\ & & & 2 & 2 \end{array}\right) w = \begin{pmatrix} \alpha \\ 0 \\ \beta \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} w_4 = w_5 = 0 \text{ as well} \\ \beta = 0 \\ w_2 = \alpha \neq 0 \end{array} \Rightarrow \begin{pmatrix} w_1 \\ 1 \\ w_3 \\ 0 \\ 0 \end{pmatrix} \text{ works fine for } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right] e^{-t}$$ is the third solution for $\lambda = -1$.

Back to $1 \pm 2i$: as before, since $1 \pm 2i$ is not an eigenvalue of $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ we can do computation for

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & 1 \end{bmatrix} : \left[\begin{array}{ccc|c} -2-2i & 1 & 1 & 0 \\ 0 & -2i & -2 & 0 \\ 0 & 2 & -2i & 0 \end{array}\right] \Rightarrow \begin{bmatrix} -2(1+i) & 1 & 1 \\ 0 & 1 & -i \\ 0 & 1 & -i \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2(\cancel{i+1})v_1 = v_2 + v_3 = iv_3 + v_3 = (\cancel{1+i})v_3 \Rightarrow \begin{pmatrix} v_3/2 \\ iv_3 \\ v_3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 2i \\ 2 \end{pmatrix} = v = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}}_{y} i$$
$$v_2 = iv_3$$

$\Rightarrow$ two solutions: $$e^t \left[\cos(2t) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}\right] ; \quad e^t \left[\cos(2t) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \sin(2t) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}\right]$$

in general $$e^{\alpha t}\left[\cos(\beta t) x - \sin(\beta t) y\right] ; \quad e^{\alpha t}\left[\cos(\beta t) y + \sin(\beta t) \vec{x}\right]$$

$\Rightarrow$ general solution $$c_1 e^t \left[\cos(2t) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}\right] + c_2 e^t \left[\cos(2t) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \sin(2t) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}\right] +$$

$$+ c_3 e^{-t}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} + c_4 e^{-t}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix} + c_5 e^{-t}\left[\begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}\right]$$

Review higher-order linear

* homogeneous one

* Use variation of parameters or undetermined coeff.s for the particular solution.

Example: $y^{(4)} + y^{(3)} = e^t + \frac{1}{t}$

* homogeneous: char. polynomial $t^4 + t^3 = 0 \Rightarrow t^3(t+1) = 0$: $t=0 \quad k=3$, $t=-1 \quad k=-1$

$\Rightarrow$ gen. sol. $c_1 + c_2 t + c_3 t^2 + c_4 e^{-t}$.

$$W = \det\begin{bmatrix}1 & t & t^2 & e^{-t}\\ 0 & 1 & 2t & -e^{-t}\\ 0 & 0 & 2 & e^{-t}\\ 0 & 0 & 0 & -e^{-t}\end{bmatrix} = -2e^{-t}$$

$g(t) = e^t + \frac{1}{t} = g_1(t) + g_2(t)$

$g_1(t) = e^t$, $g_2(t) = \frac{1}{t}$

$\uparrow$ undetermined coeff.'s $\qquad \uparrow$ variation of parameters.

$g_1(t) = e^t$: $t^s \cdot A \cdot e^t$. $s=0$ (e^t is not among the solutions for the homogeneous one)

$\Rightarrow Ae^t + Ae^t = e^t \Rightarrow Y_{p,1} = \frac{e^t}{2}$.

$g_2(t) = \frac{1}{t}$.

$$W_1 = \det \begin{bmatrix} 0 & t & t^2 & e^{-t} \\ 0 & 1 & 2t & -e^{-t} \\ 0 & 0 & 2 & e^{-t} \\ 1 & 0 & 0 & -e^{-t} \end{bmatrix} = -\left[\cancel{2}t^2 \cdot e^{-t} + 2e^{-t} \cancel{- t^2 e^{-t}} + 2te^{-t} \right]$$

$$= -e^{-t}\left[t^2 + 2t + 2 \right]$$

$$W_2 = \det \begin{bmatrix} 1 & 0 & t^2 & e^{-t} \\ 0 & 0 & 2t & -e^{-t} \\ 0 & 0 & 2 & e^{-t} \\ 0 & 1 & 0 & -e^{-t} \end{bmatrix} = 2te^{-t} + 2e^{-t} = e^{-t}(2t+2)$$

$$W_3 = \det \begin{bmatrix} 1 & t & 0 & e^{-t} \\ 0 & 1 & 0 & -e^{-t} \\ 0 & 0 & 0 & e^{-t} \\ 0 & 0 & 1 & -e^{-t} \end{bmatrix} = -e^{-t}$$

$$W_4 = \det \begin{bmatrix} 1 & t & t^2 & 0 \\ 0 & 1 & 2t & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 2.$$

$$\Rightarrow y_{p,2} = 1 \int \left(-\frac{1}{2te^{-t}} \right)(-e^{-t})(t^2+2t+2)\,dt + t \int \left(-\frac{1}{2te^{-t}} \right) e^{-t}(2t+2)\,dt + t^2 \int \left(-\frac{1}{2te^{-t}} \right)(-e^{-t})\,dt$$

$$+ e^{-t} \int \left(-\frac{1}{2te^{-t}} \right) \cdot 2\,dt$$

$\Rightarrow$ A particular solution is given by $y_{p,1} + y_{p,2}$.

Review for non-homogeneous systems: we have the formula $\Psi(t)\left[\int \Psi(t)^{-1} b(t)\,dt + K\right]$.

Example: $x' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} x + \begin{pmatrix} e^t \\ e^t \sin(2t) \end{pmatrix}$

we have computed before a fund. set of solutions for $x' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} x$ $\lambda = 1+2i$ & $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ &

$y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\Rightarrow$ $e^t\left[\cos(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin(2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$; $e^t\left[\cos(2t)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(2t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$

$$= \begin{pmatrix} -\sin(2t)e^t \\ \cos(2t)e^t \end{pmatrix} \qquad = \begin{pmatrix} \cos(2t)e^t \\ \sin(2t)e^t \end{pmatrix}$$

$$\Psi(t) = \begin{pmatrix} -\sin(2t)e^t & \cos(2t)e^t \\ \cos(2t)e^t & \sin(2t)e^t \end{pmatrix} \rightsquigarrow \Psi(t)^{-1} = \frac{1}{\det(\Psi)}\begin{pmatrix} \sin(2t)e^t & -\cos(2t)e^t \\ -\cos(2t)e^t & -\sin(2t)e^t \end{pmatrix}$$

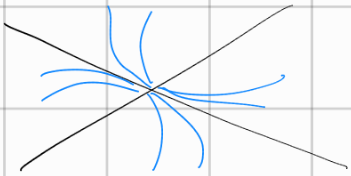
$$\det(\Psi) = -\sin^2 e^{2t} - \cos^2 e^{2t} = -e^{2t} \Rightarrow \Psi(t)^{-1} = \begin{pmatrix} -\sin(2t)e^{-t} & \cos(2t)e^{-t} \\ \cos(2t)e^{-t} & \sin(2t)e^{-t} \end{pmatrix}$$

$$\Rightarrow x(t) = \Psi(t)\cdot\left[\int \begin{pmatrix} -\sin(2t)e^{-t} & \cos(2t)e^{-t} \\ \cos(2t)e^{-t} & \sin(2t)e^{-t} \end{pmatrix}\begin{pmatrix} e^t \\ e^t\sin(2t) \end{pmatrix} dt + K\right] =$$

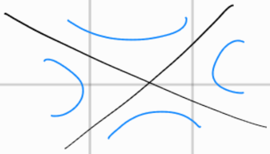
$$= \Psi(t)\left[\int \begin{bmatrix} -\sin(2t) + \cos(2t)\sin(2t) \\ \cos(2t) + \sin(2t)^2 \end{bmatrix} dt + K\right]$$

Review qualitative discussion for $x' = Ax$.

if $\lambda_1 \neq \lambda_2$ real:



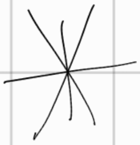
or



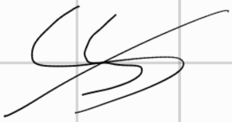
for a picture to be complete needs:

- ⊛ eigenvectors
- ⊛ arrows
- ⊛ sketch of the solutions (lines with the "right inclination") in all the regions.
- ⊛ name of the type of the equilibrium point.

if $\lambda_1 = \lambda_2$ we have

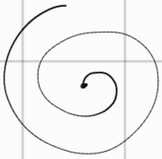


or

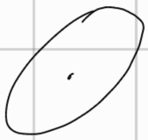


(same as above)

if $\lambda_1 = \overline{\lambda_2}$ complex:



or

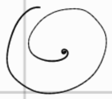
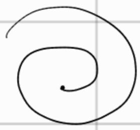


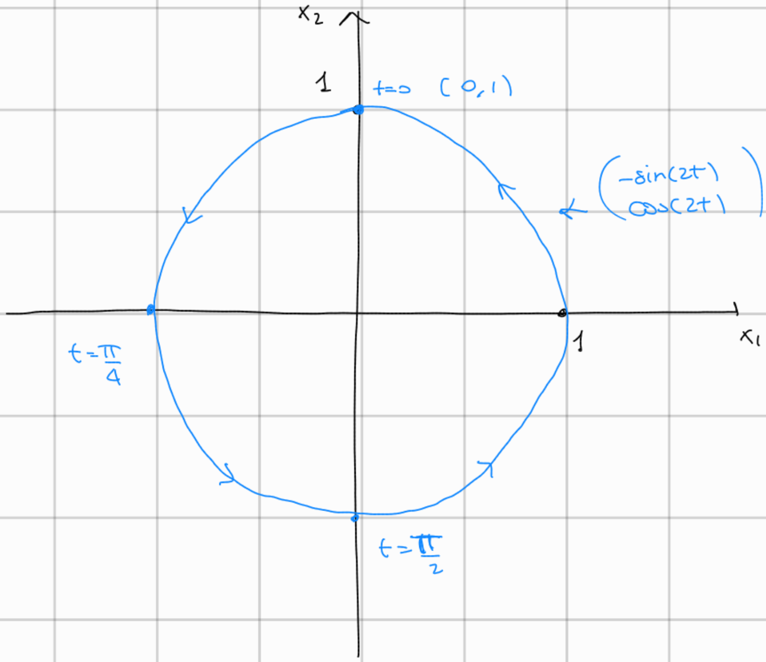
⊛ arrows ⊛ clockwise / anticlockwise ⊛ sketch

⊛ name of the point.

Example. $x' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -\sin(2t)e^t \\ \cos(2t)e^t \end{pmatrix} = y_1(t)$ $\quad y_2(t) = \begin{pmatrix} \cos(2t)e^t \\ \sin(2t)e^t \end{pmatrix}$

we are going to get a spiral point (source) $1>0$.

So enough to decide  or 



Plug values in: $y_1 = \begin{pmatrix} -\sin(2t)e^t \\ \cos(2t)e^t \end{pmatrix}$

$x_1(t) = -\sin(2t)\,e^t$

$x_2(t) = \cos(2t)\,e^t$

$$\Rightarrow \left.\begin{aligned} x_1(t)\frac{1}{e^t} &= -\sin(2t) \\ x_2(t)\frac{1}{e^t} &= \cos(2t) \end{aligned}\right\} \Rightarrow$$

$t=\frac{\pi}{4}$: $\begin{pmatrix} -\sin(\pi/2) \\ \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$t=\frac{\pi}{2}$: $\begin{pmatrix} -\sin(\pi) \\ \cos(\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

since the radius for t increasing

increases as well

$\Rightarrow$ the spiral is

counter clockwise